# Thin-plate metamaterials: physics and applications

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## ABSTRACT

A thin-plate metamaterial made of a thin plate periodically attached with mass-spring oscillators is analyzed. Based on the analytic solutions of sound waves incident on the metamaterial, effective mass density can be defined. Approximate expressions of effective mass can be derived when the first-order vibration mode of the plate is considered. It is found that the Lorentz and Drude behavior of effective mass can be obtained. As an example of potential applications, the sound insulation effects of multilayered thin-plate metamaterials are studied. High transmission loss can be achieved in a finite-layered metamaterials at negative-mass frequencies. Their applications to noise control can be anticipated.

**Keywords:** Metamaterial; Thin plate; Negative dynamic mass; Sound insulation

## 1. INTRODUCTION

Metamaterials are artificial composite materials having superior dynamic properties. In 2000, Liu et al [1] first proposed bulk elastic metamaterials with negative dynamic mass, which is composed of the soft-rubber coated lead spheres embedded in an epoxy matrix. With help of the mass-spring structures [2, 3], negative mass effect has been attributed to the translational resonances among constituents. Locally resonant behavior of building units brings metamaterials superior dynamic performances at low frequencies.

Thin-plate structures are widely used in various kinds of vehicles and industrial facilities, and are often working under the dynamic loading. It is expected that the dynamic performances of thin-plate structures can be improved if the metamaterial concept is introduced. Thin-plate metamaterials are thus proposed and may be used because of their notable advantages, such as high-strength, lightweight, and ability that common metamaterials possess. When sound waves hit thin-plate structures, acoustic responses are dominantly determined by the flexural vibrations of thin plates. Some preliminary studies [4, 5] have demonstrated that locally resonant microstructures can be designed to control flexural vibration performances of thin-plate structures. Xiao et al [4] studied a thin epoxy plate containing a periodic square array of lead discs hemmed around by rubber. Oudich el al [5] studied the model made of a square lattice of cylinder stubs deposited on a thin epoxy plate. But little work studies how thin-plate metamaterials interact with incoming sound waves.

Here a thin-plate metamaterial is designed by placing locally resonant structures periodically on a thin plate. When the resonant microstructures are represented by mass and spring structures, analytical solutions are derived for plane sound waves incident on the metamaterials at arbitrary angles. According to the homogenization method, negative effective mass of metamaterials can be achieved by either local resonances of single unit or nonlocal resonances of adjacent units. In the normal incident case, a general expression of effective mass that includes the effect of local and nonlocal resonances are given. As an example of potential applications, the sound insulation effects of multilayered thin-plate metamaterials are studied.

## 2. ANALYTIC MODEL

The analytic model is shown in Fig. 1, where a thin plate is periodically attached with a series of mass-spring resonators and placed in the air background. For plane sound waves incident on the thin plate, the full-wave solutions of reflected

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and transmitted waves, and the vibration of the plate can be solved analytically. The space-harmonic method is used to solve the motion of the periodical structure [6]. The displacement of the plate and the mass can be expressed as:

$$u_1(y,t)=\sum_{n=-\infty}^{\infty} A_n e^{-j(k_y+\frac{2n\pi}{L})y} e^{j\omega t} \tag{1}$$

$$u_2(t)=\frac{k_s}{k_s-m_m\omega^2}u_1(0,t) \tag{2}$$

where $u_1(y,t)$, $u_2(t)$ are the displacements of the plate and the mass, the coefficient $A_n$ is the complex amplitude of the plate, $k_y$ is the wave propagation constant in the $y$ direction, $L$ is the length of a unit cell, $k_s$ is the spring constant, and $m_m$ is the area density of the mass.

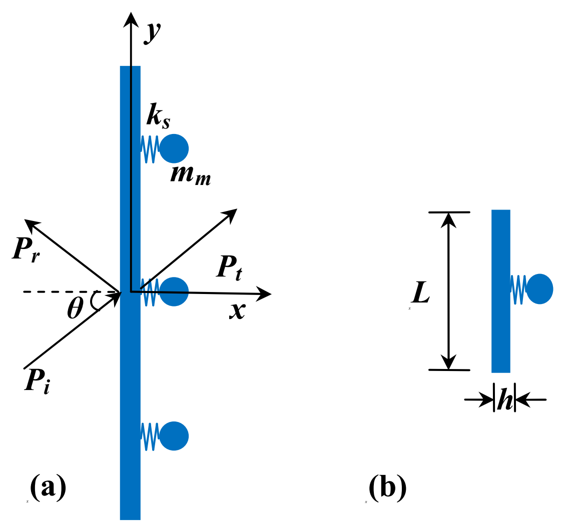



Figure 1. (a) Schematic of a thin plate attached with a series of resonators with the constant distance $L$; (b) The unit cell of the periodical structure.

The velocity potential of the air can be represented in the series form. In the left and right half-space of the plate, the velocity potentials of the air are given respectively by:

$$\Phi_1=Ie^{-j(k_x x+k_y y)}e^{j\omega t}+\sum_{n=-\infty}^{\infty}\beta_n e^{-j\left[-k_{xn}x+\left(k_y+\frac{2n\pi}{L}\right)y\right]}e^{j\omega t} \tag{3}$$

$$\Phi_2=\sum_{n=-\infty}^{n=\infty}\zeta_n e^{-j\left[k_{xn}x+\left(k_y+\frac{2n\pi}{L}\right)y\right]}e^{j\omega t} \tag{4}$$

where $k_{xn}$ is the wave number in the $x$-direction, and calculated from:

$$k_{xn}=\begin{cases}\sqrt{\left(\frac{\omega}{c}\right)^2-\left(k_y+\frac{2n\pi}{L}\right)^2} & \frac{\omega}{c}>k_y+\frac{2n\pi}{L}\\ -j\sqrt{\left(k_y+\frac{2n\pi}{L}\right)^2-\left(\frac{\omega}{c}\right)^2} & \frac{\omega}{c}<k_y+\frac{2n\pi}{L}\end{cases} \tag{5}$$

Since the thickness of plate is very small and can be neglected, the continuous condition of the normal velocity of the air and plate at the boundary $x$=0 can be expressed as:

$$-\frac{\partial \Phi_1}{\partial x} = \frac{\partial u_1}{\partial t} \tag{6}$$

$$-\frac{\partial \Phi_2}{\partial x} = \frac{\partial u_1}{\partial t} \tag{7}$$

By substituting Eqs. (1-4) into Eqs. (6) and (7), we can obtain the relationships between the coefficients of the displacement of the plate and the velocity potential of the air:

$$\beta_n = \begin{cases} I - \omega \dfrac{A_n}{k_{xn}} & n = 0 \\ -\omega \dfrac{A_n}{k_{xn}} & n \neq 0 \end{cases} \tag{8}$$

$$\zeta_n = \omega \frac{A_n}{k_{xn}} \tag{9}$$

The principle of virtual work is used to obtain the governing equations of the flexural vibration of the plate. There are three kinds of virtual work applied on the plate, which are exerted by the inertial force, the air pressure and the concentrated force produced by the mass-spring structures. So the governing equation of the plate is written as:

$$\left( D\left( k_y + \frac{2m\pi}{L} \right)^4 - m_p\omega^2 + \frac{2j\rho_0\omega^2}{k_{ym}} \right) A_m - \frac{1}{L}\frac{k_s m_m \omega^2}{k_s - m_m\omega^2} \sum_{n=-\infty}^{\infty} A_n = \begin{cases} 2j\omega\rho_0 & m = 0 \\ 0 & m = \pm1, \pm2, \pm3, \ldots \end{cases} \tag{10}$$

where $D = Eh^3(1+j\eta)/12(1-\upsilon^2)$ is the flexural stiffness of the plate, $m_p$, $E$, $\eta$, $\upsilon$, are the areal mass, Young's modulus, loss factor and Poisson's ratio of the plate respectively. The sound transmission coefficient can be obtained by:

$$\tau(\theta) = \left| \frac{I_t}{I_i} \right| = \frac{\sum_{n=-\infty}^{\infty} \left|\zeta_n\right|^2 \mathrm{Re}(k_{xn})}{\left|I\right|^2 k_{x0}} \tag{11}$$

The sound transmission loss (STL) can be calculated by:

$$STL = -10\log_{10}\tau(\theta) \tag{12}$$

## 3. ANALYSIS ON NEGATIVE EFFECTIVE MASS

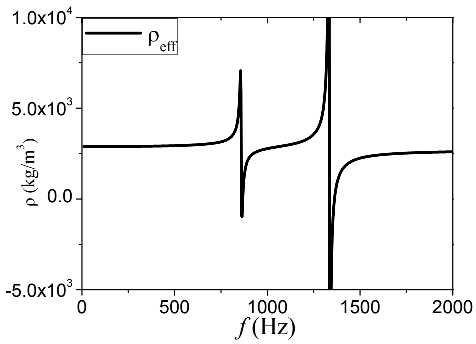
Consider the case that the wavelength is much greater than the characteristic size of the plate structure, effective dynamic mass of thin-plate metamaterials can be defined to evaluate the correlation between resonant microstructures and acoustic responses. Effective mass density is defined in a unit cell shown in Fig. 1(b) and given by [7]:

$$\rho_{\mathrm{eff}} = \frac{\overline{p}}{\langle u_{1tt} \rangle} \tag{13}$$

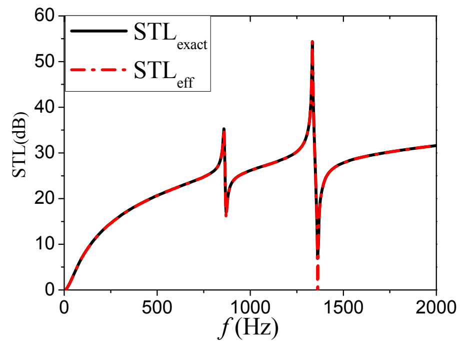
where $\overline{p}$ is the pressure integration over the left and right surfaces of the plate, and $\langle u_{1tt} \rangle$ denotes the volume integration of acceleration of the plate.

Table 1  Material and geometric parameters of thin-plate metamaterials

| | | | | | |
|---|---|---|---|---|---|
| $E$(GPa) | 70 | $\rho_p$ (kg/m$^3$) | 2700 | $h$(m) | 0.001 |
| ν | 0.3 | $m_m$ (kg) | 0.0178 | $\eta$ | 0.01 |
| $L$(m) | 0.1 | $\rho_0$ (kg/m$^3$) | 1.25 | $c$(m/s) | 343 |
| $k_s$(N/m) | $1\times10^6$ | | | | |




(a)




(b)

Figure 2 (a) Effective mass density; (b) STL of a thin-plate metamaterial and the effective homogeneous layer.

For the parameters listed in Table 1, the typical result of effective mass of a thin-plate metamaterial is shown in Fig. 2(a). The exact sound transmission loss (STL) of the plate structure is shown in Fig. 2(b), and compared with the STL of a homogeneous effective plate. It is seen that both results coincide very well, confirming the accuracy of effective dynamic mass. In Figure 2(a), there are two resonant effects corresponding to lower and higher eigen-frequency of the plate structure. It is found that the first resonance is relevant to the first-order eigenmode of the plate vibration. In the normally incident case, truncating the higher modes than $n$=1 in Eq. (1), an approximate expression of effective mass is derived as

$$\rho_{\text{eff}} = \frac{1}{Lh}\left( m_p L + \frac{m_m \omega_0^2}{\omega_0^2 - \omega^2\left(1 + 2k_s L^3 / (16D\pi^4 - L^4 m_p \omega^2)\right)} \right) \tag{14}$$

where $\omega_0{}^2 = k_s/m_m$ is the resonance frequency of the oscillator. In the following, different dynamic mass phenomena will be analyzed based on Eq. (14).

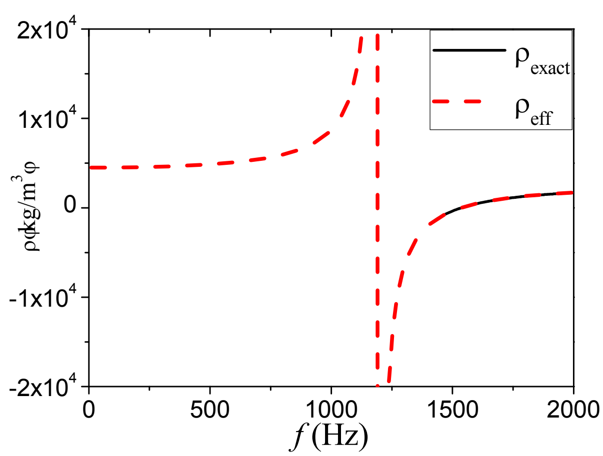



Figure 3. Exact and approximate effective dynamic mass of a thin-plate metamaterial structure.

### 3.1 Lorenz model with local resonances

When the length $L$ of the unit cell is much smaller than the wavelength, Eq. (14) can be simplified as:

$$\rho_{\text{eff}} = \frac{1}{Lh}\left(m_p L + \frac{m_m \omega_0^2}{\omega_0^2 - \omega^2}\right) \tag{15}$$

In this case, the resonance of the oscillator is dominant and the flexural vibration of the plate is very small. For the parameters in Table 1 with $L$=0.01m, figure 3 shows that the analytic expression (14) is accurate enough to predict the first resonant behavior of the thin-plate metamaterials. After the resonant frequency 1191Hz, the mass moves in the opposite direction with respect to the plate, resulting in the negative dynamic mass.

### 3.2 The Drude model

When the mass of the oscillator is set to be infinity, Eq. (14) has the Drude-form expression:

$$\rho_{\text{eff}} = \frac{1}{Lh}\left(m_p L - \frac{1}{\omega^2\left(1/k_s + 2L^3/(16D\pi^4 - L^4 m_p \omega^2)\right)}\right) \tag{16}$$

For the parameters listed in Table 1 and the stiffness of the spring is replaced by $10^7$N/m, exact and approximate effective mass densities (Fig. 4) coincide very well below the second resonance and become negative below a cut-off frequency. If the stiffness of the spring further becomes infinite, the plate metamaterials behave like a periodic array of plates with fixed boundary conditions [7]. Effective mass density can be described as:

$$\rho_{\text{eff}} = \frac{1}{Lh}\left(\frac{3}{2} m_p L - \frac{8D\pi^4}{\omega^2 L^3}\right) \tag{17}$$

The cut-off frequency is found to be $\omega_0 = \left(4\pi^2 / L^2\right)\sqrt{D/3m_p}$.

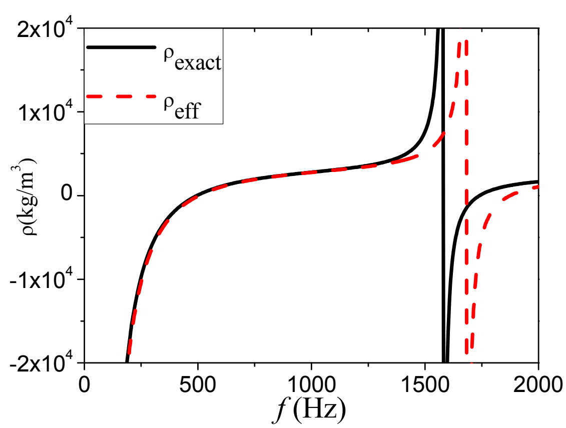



Figure 4. Exact and approximate effective dynamic mass of a thin-plate metamaterial with the infinite $m_m$.

### 3.3 Lorenz model without local resonance

In above cases, the periodicity $L$ is very small, so that the flexural vibration of the plates is very weak. When $L$ is increased and the flexural vibration is dominant, negative mass effect may still exist due to the nonlocal resonance of adjacent units. The resonator acts as a positive mass unit and the slender plate is like a spring. The physics of negative dynamic mass has been discussed by He et al [8] who studied a water-immersed soft solid plate patterned with periodical heavy gratings. If $L$ is further increased, high-order resonant vibrations could be produced, leading to multiple frequency region of negative effect mass. With enough terms contained in the displacements (1), equation (13) can give accurate prediction of effective mass.

## 4. SOUND SHIELDING BY THIN-PLATE METAMATERIALS

To analyze the blocking effects of sound waves, $N$ layers of thin-plate metamaterials separated by the constant distance $d$ are considered. Based on the analytic method of single layer metamaterials, sound transmission loss of $N$ layers can also be computed. At frequencies of negative dynamic mass, it is expected that transmission will be greatly lowered. As an example, figure 5(a) shows effective mass density of a single metamaterial plate, which becomes negative from 375Hz to 1040Hz. Figure 5(b) shows the STL of five-layer thin-plate metamaterials. In the negative-mass region, high transmission loss can be obtained. Potential applications to noise shielding can be anticipated.

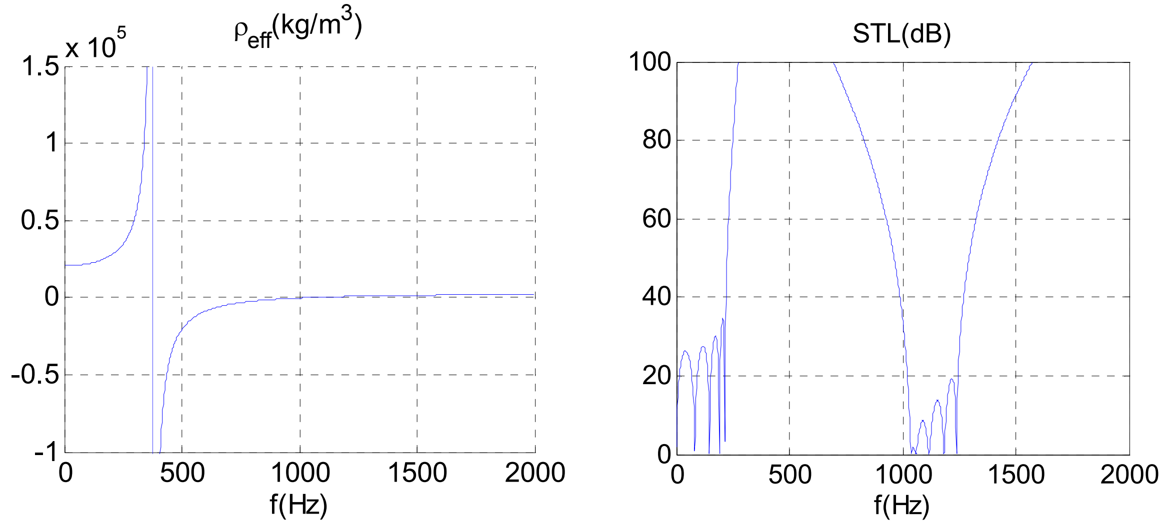



Figure 5 (a) Effective mass density of a single plate. (b) The STL of five-layer metamaterials. The distance between adjacent layers is $d$=0.01m, the length of the unit cell of each layer is $l$=0.01, the mass of the oscillator is 0.1785kg, other parameters are listed in Table 1.

## 5. CONCLUSIONS

A thin-plate metamaterial is analyzed and effective mass density can be obtained under sound wave excitations. When the length $L$ of the unit cell is very small, acoustic response of the plate metamaterial is dominated by the mass-spring resonance. With the increasing of the length $L$, the flexural deformations become remarkable, and the non-local resonances of the adjacent units dominate acoustic responses of the plate. Approximate analytic expression of effective mass can be derived when the lower order eigenmode of the plate is considered. Effective mass follows either the Lorentz or Drude model, depending on the parameters of the attached resonantors. As applications of thin-plate metamaterials, high sound transmission loss can be obtained at negative mass frequencies.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