Heterogeneous Structures with Negative Effective Mass

Shanshan Yao, Xiaoming Zhou, and Gengkai Hu

Abstract Dynamic mass of a composite is different from the gravitational one; careful placement of mass elements in the composite may result in resonance under wave loading. This resonant effect will induce negative effective mass, which has a substantial effect on the overall behavior of the composite. We first study the lattice wave propagation in a discrete mass-spring system, where mass-in-mass structures are connected by elastic springs. Around the resonant frequency, composite units are demonstrated both by theoretical and experimental methods to have negative effective mass, leading to anomalous dispersion effect. Its counterpart in continuum materials is the epoxy matrix filled with rubber-coated lead spheres, where effective mass of such a composite is shown to be negative near the resonant frequency, due to the out-of-phase motion of heavy inclusions in the composite. When the inner mass is clamped, negative effective mass is found to take place in the band below a critical frequency. For a two dimensional waveguide with rigid clamped boundary, it can be homogenized as a material with negative mass following the Drude relation. These findings will have potential applications in low-frequency noise barriers and vibration dampers.

Keywords Acoustic metamaterials • Negative effective mass • Low frequency • Drude relation

1 Introduction

Manipulating waves with materials is actually an intensive research field, especially refueled recently by the conception of metamaterials. Acoustic metamaterials are composites with carefully designed microstructures, which will resonate under

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acoustic waves; the effective mass density or modulus of the composites can be negative near the resonant frequency [1, 2]. These unusual properties can be used to design interesting devices, e.g., acoustic cloaking [3], superresolution imaging [4]. They can also be used to shield low frequency noises, which is difficult with traditional materials due to the mass law. The first acoustic metamaterial was proposed by Liu et al. [1], and the low transmission gap is attributed to negative effective mass of a lead-rubber-epoxy composite [5]. The negative mass for a mass-spring system and for a composite with coated particles is shown to result from out-of-phase motion of the particle and its surrounding [6]. However the effective mass of such structures follows the Lorentz model, i.e., it is only negative near the resonant frequency. For engineering applications, metamaterials with a broadband negative effective mass are required. To this end, recently, Lee et al. [7] show that a stretched rubber membrane with fixed outer boundary can be homogenized as an acoustic metamaterial with effective mass following the Drude relation, i.e., its effective mass is negative below a cut-off frequency. Their result paves a way for broadband applications of acoustic metamaterials, especially for the low frequency range. In this paper, we examine the mechanism of acoustic metameterials with negative mass following the Drude relation. Different structures, including a spring-mass system, composite with coated particles and two dimensional waveguide are investigated. The mechanism of achieving broadband negative mass will be demonstrated clearly. Based on these results, a new acoustic metamaterial with a lattice structure will be proposed, which can stop low frequency noise and have loading capacity as well.

2 Discrete System Composed of Mass and a Spring

2.1 Lorentz Model and Drude Model

The analyzed mass-spring system is composed of an outer mass m_0 connected inside with a hidden mass m_1 by two linearly elastic springs with equal spring coefficient G, as shown in Fig. 1a. Suppose that a harmonic force F of angular frequency ω acts on the outer mass, resulting in the displacements u and U respectively for the bodies m_0 and m_1. It is possible to define an effective mass m_{eff} to describe the macroscopic force-displacement relation $F = -m_{\mathrm{eff}}\omega^2 u$, where effective mass m_{eff} can be obtained as follows:

The equilibrium equations are

$$2G\,(u - U) = -m_1\omega^2 U \tag{1}$$

for the internal mass m_1, and

$$F - 2G\,(u - U) = -m_0\omega^2 u \tag{2}$$

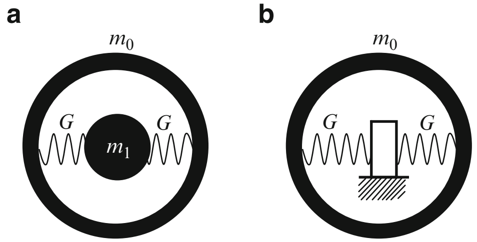

Fig. 1 Mass-spring structures with effective mass following (**a**) the Lorentz relation and (**b**) the Drude relation

for the outer mass. Combining Eqs. (1) and (2), we get the following relation

$$F = -\left(m_0 + \frac{2Gm_1}{2G - m_1\omega^2}\right)\omega^2 u \tag{3}$$

From Eq. (3), effective mass m_{eff} has the following Lorentz-type relation

$$m_{\text{eff}} = m_0\left(1 + \frac{\omega_c^2}{\omega_0^2 - \omega^2}\right) \tag{4}$$

with $\omega_{\text{c}} = \sqrt{2G/m_0}$ and $\omega_0 = \sqrt{2G/m_1}$.

It can be found from Eq. (4) that the effective mass is negative in the frequency band $\omega_0 < \omega < \sqrt{\omega_0^2 + \omega_{\text{c}}^2}$. Negative effective mass implies that the applied force gives rise to acceleration of the composite in the opposite direction. This peculiar phenomenon originates from the remarkable anti-phase motion of the internal mass with respect to the outer one, i.e., $U/u < 0$ in the negative-mass band.

Now consider the case where the inner mass m_1 is clamped, or equivalently the inner mass approaches infinity. According to Eq. (4), the effective mass of the system with clamped inner mass becomes

$$m'_{\text{eff}} = m_0\left(1 - \frac{\omega_{\text{c}}^2}{\omega^2}\right) \tag{5}$$

Equation (5) shows a different negative-mass behavior compared to that given by Eq. (4): the effective mass obeys the Drude-type relation and becomes negative below a cut-off frequency ω_{c}. Since negative mass is closely related to the wave attenuation, this property will be very useful in broadband application to low-frequency sound insulation.

2.2 Experimental Validation

To examine the wave attenuation mechanism induced by negative mass, we construct a periodic lattice system by connecting the negative-mass structures with

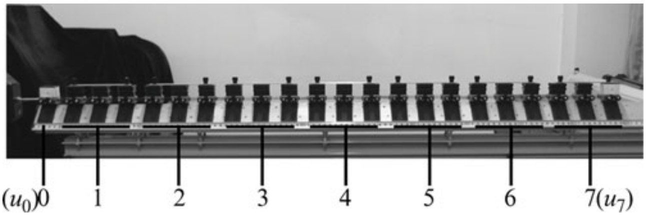

Fig. 2 The air track lifting system

springs K. Due to the periodicity, the dispersion relation of such a system is derived as

$$m_{\mathrm{eff}}\omega^2 = 4K\sin^2\frac{q_{\mathrm{s}}a}{2} \tag{6}$$

where q_{s} is the Bloch wave vector and a denotes the spacing between two adjacent elements. From Eq. (6), q_{s} will take purely imaginary values if m_{eff} is negative, leading to a bandgap effect which can be demonstrated by experiment.

The air track lifting system [6] is used to measure the transmission of vibrations through a finite period of negative-mass structures, as shown in Fig. 2. The transmittance of a seven-unit system is measured by CCD techniques. Experimental parameters are $m_1 = 46.47\,g$, $m_0 = 101.10\,g$, $G = 37$ N/m, and $K = 117$ N/m. The measured transmission amplitude of vibrations through negative-mass structures and the dispersion curves of corresponding infinite-lattice systems are shown in Fig. 3. Figures 3a, b give the results for the structures shown in Figs. 1a, b respectively. In Fig. 3a, the transmission amplitude is drastically lowered at frequencies from 5.8 to 7.6 Hz due to the negative mass behavior of single units. The transmission profile is a typical one that is induced by effective mass following the Lorentz relation. In Fig. 3b, the low transmission takes place at frequencies below 4.3 Hz, corresponding to the negative effective mass in this region. In addition, the frequency bands for low transmission phenomena correlate very well with the bandgap regions. These experimental results clearly demonstrate the existence of negative effective mass following the Lorentz and Drude relations.

3 Composite with Coated Spheres

3.1 Micromechanical Model for Effective Mass

Consider a three-phase composite with coated spheres, in the long wavelength limit, the single inclusion method can be used to estimate the effective mass of the composite. The representative volume element is a coated sphere embedded

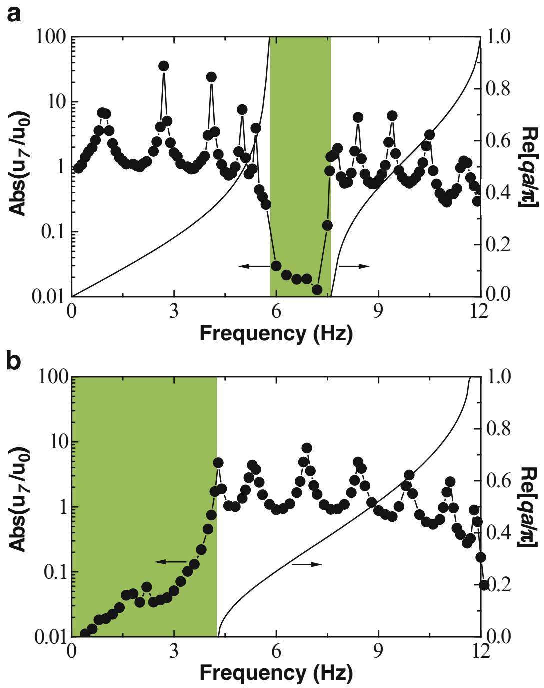

Fig. 3 Transmission amplitudes for a seven-unit system composed of (**a**) the structure shown Fig. 1a and (**b**) the structure shown Fig. 1b, as well as the corresponding dispersion curves

in an infinite matrix material. When a plane longitudinal wave is incident on the particle, the stresses have the following general expressions [8]

$$\sigma_{rr} = \frac{2}{r^2}\sum_{n=0}^{\infty} \sigma_{rr}^n P_n(\cos\theta) \tag{7a}$$

$$\sigma_{r\theta} = \frac{2}{r^2}\sum_{n=0}^{\infty} \sigma_{r\theta}^n \frac{d\,P_n(\cos\theta)}{d\theta} \tag{7b}$$

with

$$\sigma_{rr}^n = \mu\,(D_1 a_n + D_2 b_n + D_3 c_n + D_4 d_n)$$

$$\sigma_{r\theta}^n = \mu\,(E_1 a_n + E_2 b_n + E_3 c_n + E_4 d_n)$$

where $\alpha = \omega/v_l$ and $\beta = \omega/v_t$, v_l and v_t are respectively longitudinal and transverse wave velocities. $j_{\mathrm{n}}(\mathrm{x})$ and $h_{\mathrm{n}}(\mathrm{x})$ are the Bessel function and Hankel function of the first kind, respectively. Note that $c_n = (2\mathrm{n}+1)i^n$ in the matrix and all other unknown scattering coefficients a_n, b_n and d_n can be determined by continuous conditions of displacements and stresses at the boundaries. D_i and E_i are known coefficients [8].

Suppose that the radii of the sphere and coated sphere are respectively r_1 and r_2, and define the radius r_3 by $(r_2/r_3)^3 = \varphi$, where φ is the volume fraction of the coated spheres. By integrating the equation of motion, the net forces that act on the sphere, coated sphere, and matrix are given respectively by [9]

$$F_1 = \frac{8\pi}{3}\left[\sigma^1_{rr}(r_1) + 2\sigma^1_{r\theta}(r_1)\right], \tag{8a}$$

$$F_2 = \frac{8\pi}{3}\left[\sigma^1_{rr}(r_2) + 2\sigma^1_{r\theta}(r_2)\right]. \tag{8b}$$

$$F_3 = \frac{8\pi}{3}\left[\sigma^1_{rr}(r_3) + 2\sigma^1_{r\theta}(r_3)\right]. \tag{8c}$$

Then the macroscopic equations of motion are written as

$$F_1 = -\rho_1\omega^2 u_1, \quad r \le r_1, \tag{9a}$$

$$F_2 - F_1 = -\rho_2\omega^2 u_2, \quad r_1 \le r \le r_2, \tag{9b}$$

$$F_3 - F_2 = -\rho_3\omega^2 u_3, \quad r_1 \le r \le r_2, \tag{9c}$$

where ρ_1, ρ_2 and $\rho_{3;}u_1$, u_2 and u_3 are separately mass densities and volume integrated displacements along the incident direction for the sphere, cover layer and matrix. According to the general relation $F_3 = -\rho_{\text{eff}}\omega^2 u_{\text{total}}$, effective mass of the composite is defined as

$$\rho_{\text{eff}} = \frac{F_3}{F_1/\rho_1 + (F_2 - F_1)/\rho_2 + (F_3 - F_2)/\rho_3}. \tag{10}$$

3.2 Numerical Examples

As an example, a composite consisting of rubber coated lead spheres embedded in an epoxy matrix is examined. The material parameters are $\rho_1 = 11{,}600$ kg/m^3, $\lambda_1 = 42.3$ GPa, $\mu_1 = 14.9$ GPa for lead, $\rho_2 = 1{,}300$ kg/m^3, $\lambda_2 = 0.6$ MPa, $\mu_2 = 0.04$ MPa for silicone rubber, and $\rho_3 = 1{,}180$ kg/m^3, $\lambda_3 = 4.43$ GPa, $\mu_3 = 1.59$ GPa for epoxy. The volume fraction is 30 %. The sphere has a radius 5.0 mm and the coating thickness 2.5 mm. Figure 4 shows effective mass density predicted by the proposed model. When the sphere is free to move, effective mass becomes negative around the resonant frequency 400 Hz, showing the Lorentz behavior. The displacement of the core versus that of the matrix is also given in Fig. 5. It can be seen that the negative mass is induced by the resonant effect of the lead particles, which is accompanied by the abrupt change in phase. When the sphere is fixed, effective mass shows the Drude-type behavior instead and

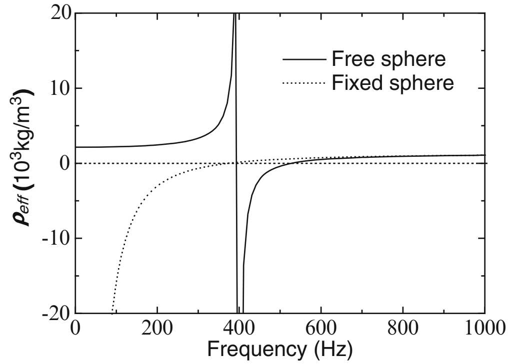

Fig. 4 Effective mass of a composite with coated sphere in the case that (**a**) the sphere is free to move and (**b**) the sphere is fixed

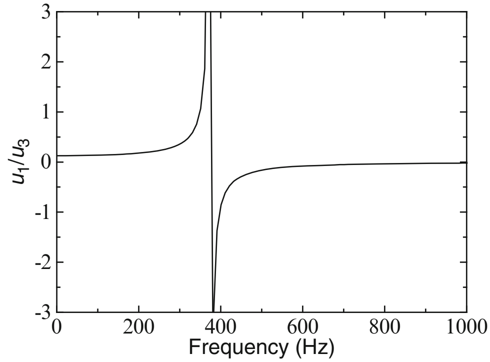

Fig. 5 The ratio of displacements between the free sphere and the matrix

becomes negative below around 400 Hz. This example demonstrates that effective mass following the Lorentz relation or Drude relation can be realized in particulate composites.

4 Two-Dimensional Waveguide with Clamped Boundary

4.1 Negative-Mass Induced Bandgap Effect

Consider a two-dimensional waveguide filled by an elastic material and infinitely extended in the out-of-plane direction. When the waveguide has clamped boundary conditions, the dispersion relations are expressed by [10]

$$\omega^2/\omega_{\mathrm{d}}^2 = m^2 + \zeta^2, \quad m = 1, 2, ... \tag{11a}$$

for P waves, and

$$\omega^2/\omega_{\mathrm{s}}^2 = n^2 + \zeta^2, \quad n = 1, 2, ... \tag{11b}$$

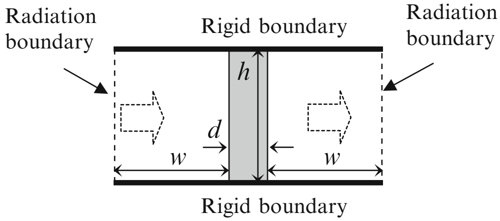

Fig. 6 The simulation environment for sound transmission of an elastic slab

for *SV* waves, where $\zeta = qh/\pi$, $\omega_{\mathrm{d}} = \pi v_{\mathrm{d}}/h$, $\omega_{\mathrm{s}} = \pi v_{\mathrm{s}}/h$, h is the width of the waveguide, v_{d} and v_{s} are respectively the wave velocities of longitudinal and shear waves, q is the propagation constant along the wave-guiding direction. For conventional elastic materials, the shear wave velocity is always less than longitudinal wave velocity, which means $\omega_{\mathrm{s}} < \omega_{\mathrm{d}}$. Therefore the dispersion relation (11) can give a lowest cut-off frequency ω_{s} for such waveguide, below which both *P* and *SV* waves are not allowed since the non-dimensional propagation constant ζ will be purely imaginary.

The bandgap effect can be attributed to the negative mass effect. Consider the general dispersion relation $(q/\omega)^2 = \rho_{\mathrm{eff}}/\mu$, where μ is the shear modulus of the filling material. Let $n = 1$ in Eq. (11b), effective mass density ρ_{eff} for the waveguide with clamped boundary condition is derived as

$$\rho_{eff} = \rho\left(1 - \frac{\omega_{\mathrm{s}}^2}{\omega^2}\right), \tag{12}$$

where ρ is the mass density of the filling material. Equation (12) should be valid below the frequency that the second branch for either *SV* or *P* modes is not involved.

To relate the bandgap effect of the waveguide to negative effective mass, sound transmission through an elastic slab with clamped boundaries will be analyzed. A plane sound wave is incident on the slab that is immersed in a fluid, as shown in Fig. 6. Effective mass density $\rho_{eff}^{'}$ for such a slab can be defined by homogenizing the equation of motion and is given by

$$\rho_{\mathrm{eff}}' = (p_1 - p_2)/a_n, \tag{13}$$

where p_1 and p_2 represent respectively the total pressures on the front and back surfaces of the slab. a_{n} denotes average accelerations of the slab. $\rho_{eff}^{'}$ can be evaluated by finite-element method (FEM).

As an example, we choose the mass density 950 kg/m^3, Young's modulus 8.88 MPa, and Poisson's ratio 0.48 for the elastic slab, as well as the mass density 1.34 kg/m^3, and wave velocity 343 m/s for the fluid material (air). The geometric parameters used in the simulation are $d = 100$ mm, $w = 400$ mm, and $h = 50$ mm. The fluid-structure coupling boundary conditions are set at the interfaces between the slab and air. Figure 7a gives the dispersion curves of the waveguide. One branch corresponding to the lowest *SV* mode is shown up, leading to the bandgap below

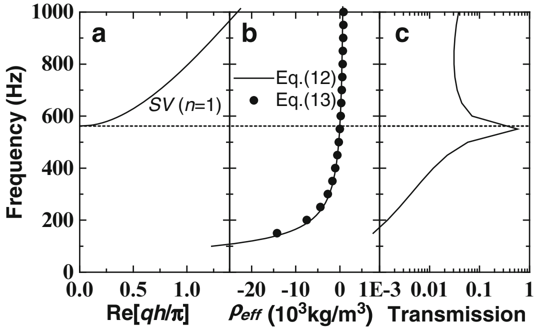

Fig. 7 (**a**) The dispersion curve of the waveguide with clamped boundaries; (**b**) effective mass density calculated by Eqs. (12) and (13); (**c**) the transmittance of sound waves through the slab obtained by FEM simulation

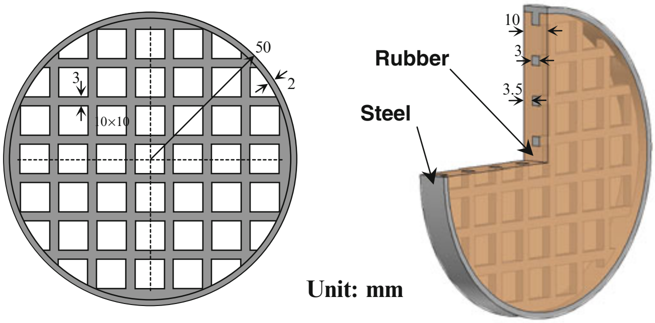

Fig. 8 Geometric sizes of designed metamaterial

562 Hz. Effective mass densities calculated by Eqs. (12) and (13) are shown in Fig. 7b. Both results coincide very well and show a negative effective mass below 562 Hz. From Figs. 7a, b, we can show that the bandgap effect of the waveguide can be indeed attributed to negative effective mass. The sound transmittance through the slab is also shown in Fig. 7c. In the bandgap region, it is found that the transmittance decreases with decreasing frequency. This effect can be explained by negative effective mass, which gives rise to the decreasing decay length at low frequencies.

4.2 Broadband Acoustic Metamaterial

Based on the previous analyses, it is shown that the waveguide with clamped boundary can stop a wave below a cut-off frequency. In this section, we will present a design and experimental validation for a broadband acoustic metamaterial with a lattice structure. The metamaterial is made of a steel grid filled with the SBR (Styrene Butadiene Rubber), as shown in Fig. 8 for the illustration. Since the lattice is much more rigid than the filled rubber, it can be considered as

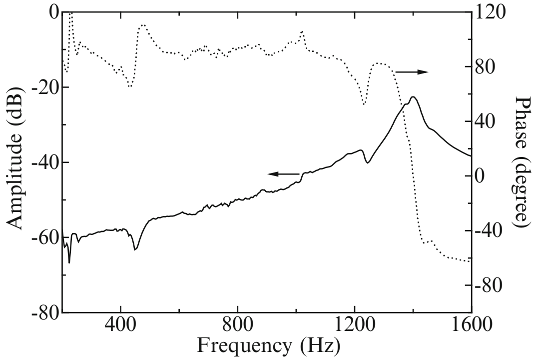

Fig. 9 Amplitude and phase of sound transmission through the metamaterial sample

periodically clamped waveguides. We can change the size of the lattice to design the cut-off frequency. To implement measurements using Brüel and Kjær type-4206T impedance tube, the grid has a circular outer shape of radius 50 mm. For enhancing the cut-off frequency, the square holes of side 10 mm are drilled with spacing 3 mm; the cut-off frequency is estimated to be round 1,400 Hz. The configuration can be considered as a periodic array of square waveguides, sandwiched by two rubber layers of thickness 3.5 mm.

The transmission of the sample is measured with a B&K 4206T impedance tube. Experimental result for sound transmission is shown in Fig. 9. A transmission peak is observed at around 1,400 Hz. The overall trend is very similar to that shown in Fig. 7c, indicating the Drude-type relation for the effective mass. The proposed acoustic metamaterial may have important implications for low frequency noise shielding, since we can use thin materials to stop low frequency noise, this is usually believed to be impossible for traditional materials due to the mass law.

5 Conclusion

We have examined different structures which can stop vibration or noise below a cut-off frenquency. For a spring-mass sysetm, it is shown that when the inner mass is clamped the effective mass follows the Drude relation, i.e., it is negative below a specific frequcency. The same idea can also be applied to composites with coated spheres, and we show that the effecive mass of the composite follows also a Drude relation if the particle is fixed or the mass of the paprticle tends to infinity. Furthermore, for a two-dimensional waveguide with clamped boundary, the bandgap of the waveguide is related to effective negative mass, which is found to follow the Drude relation. Based on these observations, we also proposed a lattice structure filled with rubber, if the lattice is made of rigid materials, the lattice structure can be considered as periodically placed small waveguides with clamped boundaries. It can stop noise below a cut-off frequency, as demonstrated experimentally.

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