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A METHOD OF PLASTICITY FOR GENERAL ALIGNED SPHEROIDAL VOID OR FIBER-REINFORCED COMPOSITES

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Abstract—Based on the general concept of the secant moduli method, together with a new way of evaluating the average matrix effective stress originally proposed by Qiu and Weng ("A Theory of Plasticity for Porous Materials and Particle-Reinforced Composites," *ASME J. Appl. Mech.* (1992), **59**, 261.), a method for nonlinear effective properties of general aligned fiber or void composites is proposed. The method is capable of predicting composite (especially for porous materials) yielding under a hydrostatic load. Compared to the Tandon and Weng ("A Theory of Particle-Reinforced Plasticity," *ASME J. Appl. Mech.* (1988), **55**, 126.), model the proposed method always gives softer prediction in the uniaxial tension. The proposed method will predict the same nonlinear stress and strain relation as the Ponte Castaneda ("The Effective Mechanical Properties of Nonlinear Isotropic Composite," *J. Mech., Phys. Solids* (1991), **39**, 45.) variational model if the same estimates or bounds for the linear comparison composite are adopted.

I. INTRODUCTION

This paper is concerned with the determination of the overall elastoplastic behavior for a class of unidirectionally aligned composites. The homogeneously dispersed inclusions (void or fiber) are assumed to be spheroidal in shape, and remain elastic. The matrix is ductile and can undergo a plastic deformation. The theoretical analyses on such problems have been performed by many authors. For example, Tvergaad [1990] and Bao *et al.* [1991] performed numerical analyses with periodic micro structures; Tandon and Weng [1988] proposed a secant moduli method for composite materials. This method makes use of a linear comparison material, whose elastic moduli at every instant are chosen to coincide with the average secant moduli of the matrix. Adopting a linear comparison material to construct the corresponding bounds or estimates for nonlinear composites has also been proposed by Willis [1991] and Ponte Castaneda [1991] (PC) through variational methods. Debotton and Ponte Castaneda [1993] applied the PC variational method to construct the stress and strain relation for nonlinear composites.

As remarked by Qiu and Weng [1992], the original secant moduli method proposed by Tandon and Weng [1988] cannot yield correct prediction for porous materials under a hydrostatic pressure. Since the average matrix effective stress is obtained directly from the average stress of the matrix, the local stress variation is smoothed out. Recently Qiu and Weng [1992] advanced an improved version of the Tandon and Weng [1988] model; they redefined the effective stress from an energy approach and evaluated it approximately.

In this paper, we will adopt the general concept of the secant moduli method, and also adopt the average matrix effective stress directly from the average of elastic distortional energy of the matrix (same as Qiu and Weng [1992]). This average matrix effective stress will be evaluated precisely by a new approach to be developed later. The emphasis will also be placed on the comparison between the proposed method and the original secant moduli model. In the end, the theoretical connection between the proposed model and the Ponte Castaneda [1991] variational method will be examined.

II. EFFECTIVE PROPERTIES OF NONLINEAR COMPOSITE MATERIALS

II.1 *Preliminary*

The considered composite is supposed to consist of two isotropic phases; $\mathbf{M}_0$, $\mathbf{M}_1$ denote the matrix and the inclusion phase compliance tensors. The inclusion phase is assumed to be elastic and spheroidal in shape, with x_1 being the symmetry axis; c_0, c_1 are the volume fractions of the matrix and the inclusion phases, respectively.

Elastic analysis of composite effective properties can be concentrated on the representative volume element (RVE). On its boundary, if a uniform stress Σ is prescribed, it was shown that the average local stress over the RVE is equal to Σ. If the average local strain over the RVE is noted by $\mathbf{E}$, we have

$$\mathbf{E} = \mathbf{M}_c \Sigma \tag{1}$$

Equation (1) gives the elastic stress–strain relation of the composite material. $\mathbf{M}_c$ is the composite compliance tensor; it depends on the phase properties and the microstructural distribution. Evaluation of $\mathbf{M}_c$ needs detailed material phase distribution information, which is almost impossible analytically. However, bounds or estimates on $\mathbf{M}_c$ can be performed. For example, based on the Mori–Tanaka mean field theory, the estimation on the composite compliance tensor can be written as

$$\mathbf{M}_c = \mathbf{M}_0 + c_1 [(\mathbf{M}_1 \mathbf{M}_0^{-1} - \mathbf{I})^{-1} + c_0 (\mathbf{I} - \mathbf{S})]^{-1} \tag{2}$$

where $[\mathbf{A}]^{-1}$ is the inverse of the tensor $\mathbf{A}$; $\mathbf{I}$ is the unit tensor; $\mathbf{S}$ is the Eshelby tensor (the detail analytical form can be found in Mura [1987]).

Weng [1992] have shown that the composite stiffness evaluated by (2) corresponds to the Willis [1977] lower bound if the matrix is the softer phase or upper bound inversely.

When the matrix undergoes a plastic deformation (the inclusion phase remains always elastic), we will use the following strain potential to characterize its elastoplastic deformation,

$$\psi = \varphi(\sigma_e) + \frac{1}{2k_0}\sigma_m^2 \tag{3}$$

The matrix is assumed plastically incompressible; σ_e and σ_m are the von Mises stress and the hydrostatic stress defined by: $\sigma_e = (3/2\ \sigma^1 : \sigma^1)^{1/2}$; $\sigma_m = 1/3 tr\sigma$.

The stress and strain relation of the matrix can be obtained by

$$\varepsilon_e = \varphi^1(\sigma_e)$$
$$\varepsilon_m = \frac{1}{k_0}\sigma_m \tag{4}$$

where ε_e is the effective strain and $\varepsilon_e = (2/3\ \varepsilon':\varepsilon')^{1/2}$; $\varepsilon_m = 1/3 tr\varepsilon$.

The secant shear modulus μ_0^s and secant bulk modulus k_0^s of the matrix are given by

$$\mu_0^s = \frac{\sigma_e}{3\varphi'(\sigma_e)} \qquad k_0^s = k_0 \tag{5}$$

The secant moduli method (Tandon *et al.* [1988]) for composite plasticity indicates that under a macroscopic applied load Σ, which corresponds to the deformation state of the matrix σ_e, the matrix can be considered as a linear isotropic material characterized by the matrix secant moduli (5), with a corresponding compliance tensor denoted by $\mathbf{M}_0^s$. By this manipulation, (2) still holds, even when the matrix enters the plastic range. In this case, the composite compliance tensor is understood as the secant composite compliance tensor $\mathbf{M}_c^s$, depending on the deformation state of the matrix through $\mathbf{M}_c^s$. If the relation between the applied macroscopic load Σ and the matrix deformation state parameter σ_e is given, the nonlinear stress and strain relation of the composite can then be derived.

In what follows, we will focus on establishing the relation between the applied macroscopic stress Σ and the effective stress in the matrix σ_e. When the matrix enters the plastic range, we will still keep he notation $\mathbf{M}_c$ and $\mathbf{M}_c$ for the matrix and the composite secant compliance tensors.

II.2 *Average stress in matrix and inclusion*

For the RVE under a uniform applied macroscopic stress Σ on its boundary (the corresponding composite strain is noted by $\mathbf{E}$), we have:

$$\Sigma = c_0 < \sigma >_0 + c_1 < \sigma >_1 \tag{6}$$

$$\mathbf{E} = c_0 < \varepsilon >_0 + c_1 < \varepsilon >_1 \tag{7}$$

where $<\mathbf{A}>_0$ is the volume average over the matrix of the quantity $\mathbf{A}$, $<\mathbf{A}>_1$ is that over the inclusion phase.

With the help of the phase constitutive relations, (7) can be rewritten as

$$\mathbf{E} = c_0\, \mathbf{M}_0 < \sigma >_0 + c_1\, \mathbf{M}_1 < \sigma >_1 \tag{8}$$

From (6) and (8) we can readily obtain the average stresses in the matrix and the inclusion:

$$< \sigma >_0 = \frac{1}{c_0} (\mathbf{M}_1 - \mathbf{M}_0)^{-1} (\mathbf{M}_1 - \mathbf{M}_c)\Sigma \tag{9}$$

$$< \sigma >_1 = \frac{1}{c_1} (\mathbf{M}_0 - \mathbf{M}_1)^{-1} (\mathbf{M}_0 - \mathbf{M}_c)\Sigma \tag{10}$$

Now, we are ready to recall the von Mises equivalent stress of the matrix defined in Tandon and Weng [1988]:

$$\sigma_e^2 = 3/2 < \sigma' >_0 : < \sigma' >_0 \tag{11}$$

Equations (11) together with (9) give the relation between the macroscopic applied stress Σ and the matrix effective stress σ_e. So, for a given loading type (e.g. uniaxial Σ_{11} or hydrostatic Σ loading) at a given value of the matrix effective stress σ_e, the corresponding secant matrix compliance tensor and the secant composite compliance tensor can be calculated from (5) and (2). The macroscopic load (e.g. Σ_{11} or Σ) corresponding to the given matrix deformation state σ_e can be evaluated from (9) and (11). Finally, the composite strain is obtained from (1). By increasing the value of σ_e and repeating the previous process, the entire stress and strain curve of the nonlinear composite can be derived.

This procedure is proposed for composite materials by Tandon and Weng [1988] who reformulated in a different form.

As remarked by Qiu and Weng [1992], the definition of the matrix effective stress directly from the average stress cannot take into account the local stress variation in the matrix. In fact, from (9), we observe that for isotropic phases with spherical inclusions, the composite compliance tensor is also isotropic, and thus under a pure hydrostatic load, the average stress in the matrix is also hydrostatic in nature. In this case the composite can never yield with the previous definition of the matrix effective stress (11). Thus, in the following section, a new effective stress of the matrix will be proposed.

II.3 *Average matrix second order stress moment*

We will follow the general method proposed by Bobeth and Diener [1986,1987]. In the elastic case, for the composite RVE under a constant macroscopic applied load Σ, the average stored energy of the RVE can be expressed as

$$\frac{1}{2} < \sigma.\varepsilon > = \mathbf{U} = \frac{1}{2}\Sigma\!: \mathbf{M}_c\!: \Sigma \tag{12}$$

$\mathbf{M}_c$ is the composite compliance tensor.
Equation (12) can be rewritten as:

$$< \sigma.\mathbf{M}\!:\!\sigma > = \Sigma\!: \mathbf{M}_c\!: \Sigma \tag{13}$$

where $\mathbf{M}$ is the local phase compliance tensor.

Now, under a constant applied macroscopic load Σ, a variation of the local compliance tensor $\delta\mathbf{M}$ will lead to a variation of the local stress $\delta\Sigma$, which in turn will lead to a variation of the composite compliance tensor $\delta\mathbf{M}_c$. We have

$$\Sigma\!: \delta\mathbf{M}_c\!: \Sigma = < \sigma.\delta\mathbf{M}:\sigma > +2 < \sigma.\mathbf{M}:\delta\sigma > \tag{14}$$

Since, under a constant applied stress, the volume average of the local stress variation $<\delta\sigma>$ vanishes, with the help of Hill's condition (Hill [1963]), it can be shown that the second term of (14) vanishes. So we obtain

$$\Sigma\!: \delta\mathbf{M}_c\!: \Sigma = < \sigma.\delta\mathbf{M}:\sigma > \tag{15}$$

For a two-phase composite with isotropic phases, if we let the shear modulus of the matrix undergo a variation, we have

$$c_0 < \sigma':\sigma' >_0 \delta(\frac{1}{2\mu_0}) = \Sigma\!: \delta\mathbf{M}_c\!: \Sigma \tag{16}$$

Now, we follow Qiu and Weng's [1992] definition of effective stress from the average of the elastic distortional energy of the matrix, which can be evaluated from the variation of the effective compliance with respect to the variation of the local shear modulus, as

$$\sigma_e^2 = 3/2 < \sigma' : \sigma' >_0 = \Sigma : \left(-\frac{3\mu_0^2}{c_0}\frac{\delta \mathbf{M}_c}{\delta \mu_0}\right) : \Sigma \tag{17}$$

The expression of the composite compliance can be estimated or bounded. Since we can also choose (2) for an estimation, $\delta \mathbf{M}_c/\delta \mu_0$, σ_e can be evaluated without much difficulty. The Eshelby tensor **S** depends also on μ_0 through the Poisson ratio of the matrix. It is easy to check that for the isotropic phases and spherical inclusions, even under a pure hydrostatic load, the effective stress will not vanish. So we believe that the new definition of the matrix effective stress can take into account (to some extent) the local stress variation of the matrix.

With the aid of the secant moduli method (Tandon and Weng [1988]) and the above matrix effective stress defined by (17), the stress and strain relation of the nonlinear composite can then be constructed following the same procedure explained in Section II.2.

In the following section, we will analyze in detail the effective behavior of the composite predicted by the proposed method.

II.4 *Numerical applications*

The matrix is assumed to have a power law type hardening, expressed by the following:

$$\sigma_e = \sigma_y + h\varepsilon_{ep}^n \tag{18}$$

The strain potential for the matrix is

$$\psi = \frac{1}{6\mu_0}\sigma_e^2 + \frac{n}{(n+1)}\frac{1}{h^{1/n}}(\sigma_e - \sigma_y)^{\frac{n+1}{n}} + \frac{1}{2k_0}\sigma_m^2 \tag{19}$$

The secant shear modulus and the secant bulk modulus of the matrix are given by

$$\mu_0^s = \frac{1}{\dfrac{1}{\mu_0} + \dfrac{3\varepsilon_{ep}^n}{\sigma_y + h\varepsilon_{ep}^n}} \tag{20}$$

$$k_0^s = k_0 \tag{21}$$

The matrix is 6061 aluminum, with material constants of E_0 = 68.3 GPa; ν_0 = 0.33; σ_y = 250 MPa; h = 173 MPa, n = 0.455. For the reinforced phase, E_1 = 490 GPa; ν_1 = 0.17.

Figure 1 shows the contribution of pure hydrostatic pressure on the matrix effective stress defined by the first and second order stress moment ((11) and (17)) as a function of the aligned fiber aspect ratio. For spherical inclusions, this contribution is zero, based on the matrix average stress. The contribution based on the second-order stress moment is larger than that based on the matrix average stress. For voided materials, under a hydrostatic load, the average stress of the matrix is always hydrostatic in nature whatever the form of the voids (6), so the contribution on the matrix effective stress based the matrix average stress is zero. However, the contribution on the matrix

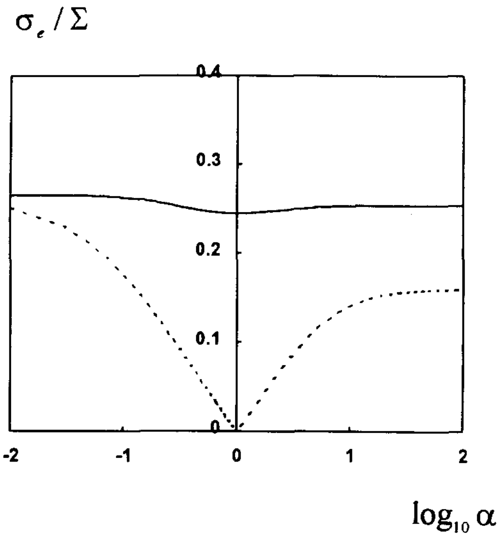

Fig. 1. The contribution of a pure hydrostatic stress on the matrix effective stress as a function of reinforced phase aspect ratios. Solid line, new model; dashed line, average stress.

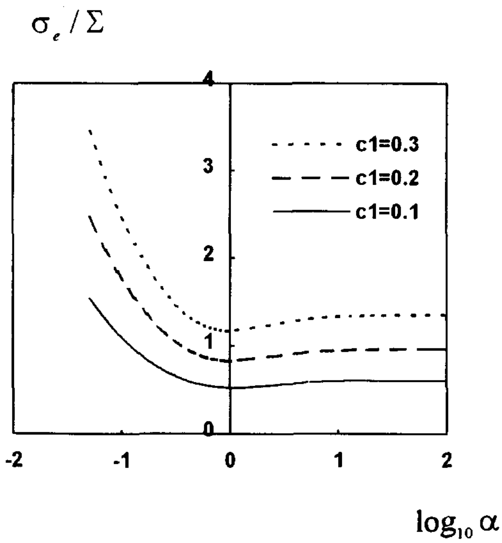

Fig. 2. The contribution of a pure hydrostatic stress on the matrix effective stress of the porous materials as a function of void aspect ratios.

effective stress based on the second-order stress moment keeps a finite value: it increases as a function of the void volume fraction (Fig. 2). For disk-type voids, this contribution is more significant.

Figures 3 and 4 show the predicted uniaxial stress and strain curves for the aligned fiber reinforced composites and porous materials. It is observed that the stress and strain predicted by the proposed method is always softer, compared with the method based on the matrix average stress. The difference becomes more significant for disk-

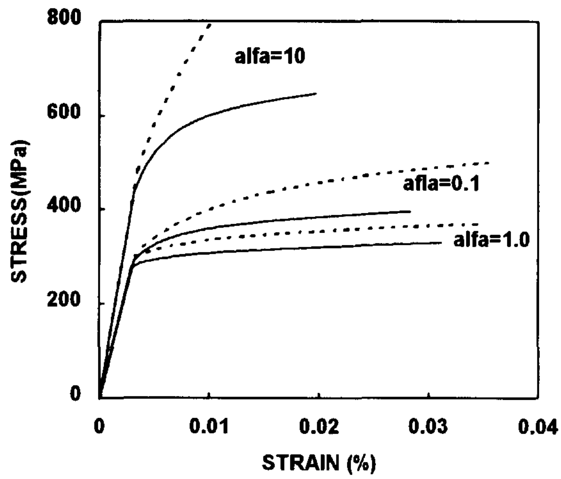

Fig. 3. Stress and strain curves of composites predicted by the proposed model (solid line) and with the model based on the average stress (11) ($c_1 = 0.2$).

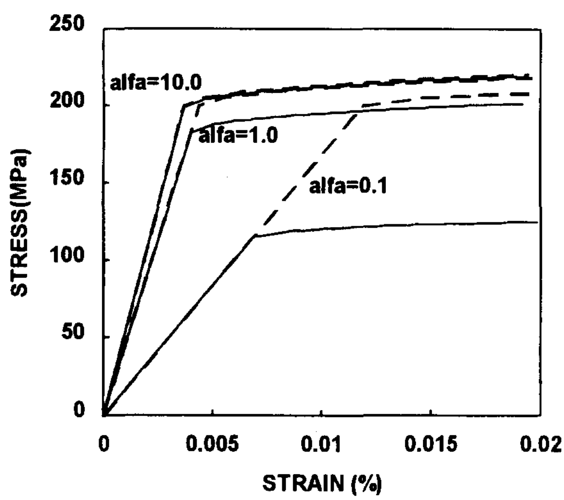

Fig. 4. Stress and strain curves of porous materials predicted by the proposed model (solid line) and with the model based on the average stress (11) ($c_1 = 0.2$).

type voids or fibers with medium aspect ratio. If we examine the contribution predicted by the two methods on the matrix effective stress under a uniaxial tension, we observe that for long cylindrical voids the matrix effective stresses are almost the same given by the two methods, but the difference becomes important for oblate voids. For the studied composite, the effective stresses of the matrix predicted by the two methods are almost the same for disk shape inclusion or long fibers. The difference is large with fibers of medium aspect ratio; this difference increases during the matrix plastic deformation (Fig. 5, curves C).

In order to examine the accuracy of the proposed method, we then apply the proposed model to the material with cylindrical void under biaxial loading $\bar{\Sigma} = \Sigma_{22} = \Sigma_{33}$,

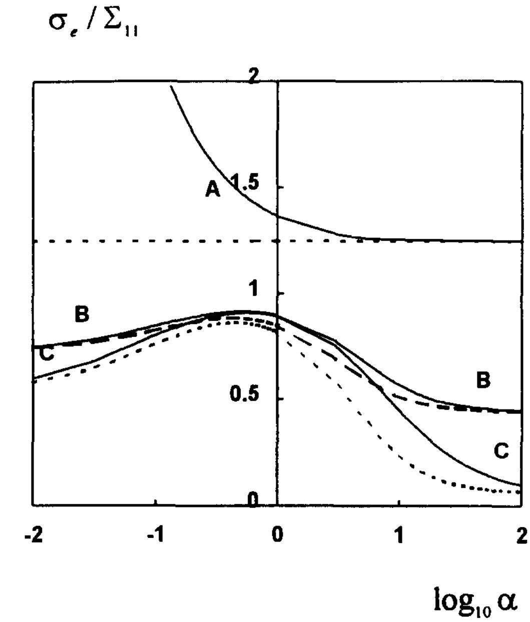

Fig. 5. The contribution of a pure tensile stress on the matrix effective stress as a function of reinforced phase aspect ratio: solid line, new model; dashed line, average stress (A, porous; B, $\mu_1/\mu_0 = 8{\cdot}16$; C, $\mu_1/\mu_0 = 8{\cdot}16$ ($c_1 = 0.2$).

$\bar{\mathrm{E}} = \mathrm{E}_{22} = \mathrm{E}_{33}$ (plane strain condition), and the material with spherical voids under a hydrostatic loading $\bar{\Sigma} = \Sigma_{kk}$, $\bar{\mathrm{E}} = \mathrm{E}_{kk}$, since in these cases the exact local analyses can be performed (for further details, we refer the reader to Qiu & Weng [1992,1993]). For the matrix $n = 1.0$; $E_p/E_0 = 0.1$; and $h = 1/(1/E_p - 1/E_0)$, the other constants remain the same. The results are shown in the Fig. 6; a good general agreement is observed for the two cases considered between the proposed method and the local exact analyses. Since the proposed method is also a mean field one, it cannot account for the detailed local yielding of the composite, which explains the difference observed near the beginning of the composite local yielding.

III. THEORETICAL CONNECTION WITH PONTE CASTANEDA VARIATIONAL APPROACH

Another approach in the literature for predicting the composite nonlinear behavior is proposed by Ponte Castaneda [1991]. The method characterizes the effective energy potential of the nonlinear composite in terms of the corresponding energy potential for the linear composite with the same microstructural characterization. This method is applied by Debotton and Ponte [1993] to construct the stress and strain relation for nonlinear composites. In this section, we will examine in detail the connection between this variational method and the proposed one in Section II.3.

For the composite with isotropic phases, the inclusion remains elastic and the non-linear matrix is characterized by the strain potential given by (3). The effective energy function of the composite can be expressed as (for further details, we refer the reader

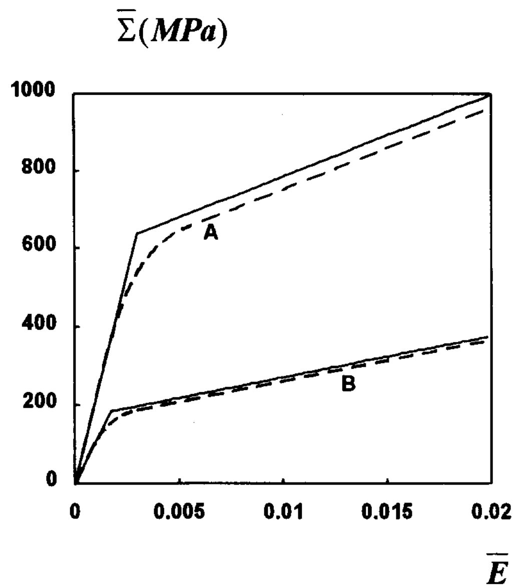

Fig. 6. Comparison between the results predicted by the present model (solid line) and the exact local solution (dashed line) for porous material, $\nu_0 = 1/2$. $c_1 = 0.3$. A, cylindrical voids under biaxial loading (plane strain); B, spherical void under a pure pressure.

to Ponte Castaneda [1991] and Debotton *et al.* [1993])

$$\bar{U} = \max_{\hat{\mu}_0 \hat{k}_0 > 0} [U_0 - \max_{\sigma_e, \sigma_m}[c_0(\frac{1}{6\hat{\mu}_0}\sigma_e^2 + \frac{1}{2\hat{k}_0}\sigma_m^2 - \varphi(\sigma_e) - \frac{1}{2k_0}\sigma_m^2)]] \tag{22}$$

where $\hat{\mu}_0$, $\hat{k}_0$ are the arbitrary linear comparison moduli of the matrix, and

$$U_0(\Sigma) = \min_{\sigma \in s(\Sigma)} \int_V U(\sigma) \mathrm{d}V \tag{23}$$

$U_0(\Sigma)$ is the effective energy function of the linear comparison composite with the local energy density $U(\sigma)$. The linear comparison composite consists of the isotropic matrix with the shear and bulk moduli $\hat{\mu}_0$, $\hat{k}_0$, and it has the same inclusion properties and geometry as the studied composite. $s(\Sigma)$ is the set of statistically admissible stress field in the RVE with a uniform applied stress Σ on the RVE boundary.

The effective energy function of the linear comparison composite can be bounded or estimated, and this leads to the corresponding bounds or estimates for the effective energy function of the composite. In our case, we still use (2) to estimate the linear comparison composite compliance tensor, but here the matrix shear and bulk moduli are $\hat{\mu}_0$, $\hat{k}_0$. We still use $\mathbf{M}_c$ to note the compliance tensor for the linear comparison composite. The effective energy function of the linear comparison composite is given by

$$U_0(\Sigma) = \frac{1}{2}\Sigma : \mathbf{M}_c : \Sigma \tag{24}$$

Now we will perform the two optimization processes that appear in (22).

The first optimization process inside of the brackets yields

$$\frac{1}{3\hat{\mu}_0}\sigma_e - \varphi'(\sigma_e) = 0$$
$$\frac{1}{\hat{k}_0}\sigma_m - \frac{1}{k_0}\sigma_m = 0 \tag{25}$$

or

$$\hat{\mu}_0 = \frac{\sigma_e}{3\varphi'(\sigma_e)} \tag{26}$$
$$\hat{k}_0 = k_0$$

It can be seen that (26) are identical to (5); this implies that $\hat{\mu}_0$, $\hat{k}_0$ are in fact the secant shear and bulk moduli of the matrix at the deformation state σ_e. We note that $\hat{\sigma}_e$ is the solution of (26), which depends on $\hat{\mu}_0$. So the effective energy function can be expressed as

$$\bar{U} = \max_{\hat{\mu}_0>0}[F] = \max_{\hat{\mu}_0>0}[\frac{1}{2}\Sigma\!:\!\mathbf{M}_c\!:\!\Sigma - c_0(\frac{1}{6\hat{\mu}_0}\hat{\sigma}_e^2 - \varphi(\hat{\sigma}_e)] \tag{27}$$

The optimization leads to the following equation:

$$\frac{1}{2}\Sigma\!:\!\frac{\partial \mathbf{M}_c}{\partial \hat{\mu}_0}\!:\!\Sigma + c_0\frac{1}{6\hat{\mu}_0^2}\hat{\sigma}_e^2 - c_0\frac{1}{3\hat{\mu}_0}\hat{\sigma}_e\frac{\partial \hat{\sigma}_e}{\partial \hat{\mu}_0} + c_0\frac{\partial \varphi(\hat{\sigma}_e)}{\partial \hat{\sigma}_e}\frac{\partial \hat{\sigma}_e}{\partial \hat{\mu}_0} = 0 \tag{28}$$

Using (25), we obtain

$$\hat{\sigma}_e^2 = \Sigma\!:(-\frac{3\hat{\mu}_0^2}{c_0}\frac{\partial \mathbf{M}_c}{\partial \hat{\mu}_0})\!:\!\Sigma \tag{29}$$

This matrix effective stress is exactly the effective stress defined by (17). The stress and strain relation of the nonlinear composite material can be obtained by

$$\mathbf{E}\,\frac{\partial \bar{U}}{\partial \Sigma} = \mathbf{M}_c(\hat{\mu}_0)\!:\!\Sigma + \frac{\partial F}{\partial \hat{\mu}_0}\frac{\partial \hat{\mu}_0}{\partial \Sigma} \tag{30}$$

From the optimization process (27), $\frac{\partial F}{\partial \hat{\mu}_0} = 0$, and we obtain

$$\mathbf{E} = \mathbf{M}_c(\hat{\mu}_0)\!:\!\Sigma \tag{31}$$

This equation gives the stress and strain relation of the nonlinear composites. The parameter $\hat{\mu}_0$ is determined by (26) and (29); they are exactly the same as in the proposed method in Section II.3. So, for the considered composite (the matrix yielding is pressure insensitive and follows the von Mises flow rule), both models predict the same initial yield surface and the stress–strain relation for the nonlinear composite. The matrix effective stress defined by PC's variational method corresponds exactly to the average of the local matrix effective stress.

It can be concluded that the secant moduli method with the effective stress defined from the matrix average second-order stress moment always gives the same stress and strain predictions for nonlinear composites as by PC's variational method.

IV. CONCLUSIONS

In this paper, an average matrix effective stress for linear composites is defined directly from the average of the elastic distortional energy of the matrix; i.e. the average of the local matrix effective stress. It is evaluated precisely, provided that the estimates or bounds on the linear composite compliance are known. With the aid of secant moduli method combined with the proposed matrix effective stress, a method is proposed to construct the stress and strain relations for the nonlinear composites. The method is capable of accounting to some extent for the local stress variation in the matrix and gives reasonable predictions for porous material under a pure hydrostatic loading. Compared to the secant moduli based on the average matrix stress, the proposed method always gives softer predictions in the case of uniaxial loading; the differences are larger for oblate voids or for fibers with medium aspect ratios.

It is shown theoretically that the matrix effective stress defined by PC's variational method is exactly equal to the newly defined matrix effective stress. Both methods give the same predictions for the stress and strain relations if the same estimates or bounds on the linear comparison composite are used.

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