Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
International Journal of Plasticity 21 (2005) 777–799
INTERNATIONAL JOURNAL OF Plasticity
www.elsevier.com/locate/ijplas

A continuum micromechanical theory of overall plasticity for particulate composites including particle size effect

Xiaoning Liu, Gengkai Hu *

Department of Applied Mechanics, Beijing Institute of Technology, No. 7 BaiShiQiao Road, Beijing 100081, China

Received in final revised form 27 April 2004
Available online 27 July 2004

Abstract

Classical continuum micromechanics cannot predict the particle size dependence of the overall plasticity for composite materials, a simple analytical micromechanical method is proposed in this paper to investigate this size dependence. The matrix material is idealized as a micropolar continuum, an average equivalent inclusion method is advanced and the Mori–Tanaka's method is extended to a micropolar medium to evaluate the effective elastic modulus tensor. The overall plasticity of composites is predicted by a new secant moduli method based on the second order moment of strain and torsion of the matrix in a framework of micropolar theory. The computed results show that the size dependence is more pronounced when the particle's size approaches to the matrix characteristic length, and for large particle sizes, the prediction coincides with that predicted by classical micromechanical models. The method is analytical in nature, and it can capture the particle size dependence on the overall plastic behavior for particulate composites, and the prediction agrees well with the experimental results presented in literature. The proposed model can be considered as a natural extension of the widely used secant moduli method from a heterogeneous Cauchy medium to a micropolar composite.

Keywords: Micromechanics; Particulate composite; Plasticity; Micropolar; Size effect

* Corresponding author. Tel./fax: +86-1068912731.
E-mail address: hugeng@public.bta.net.cn (G. Hu).

0749-6419/$ - see front matter
doi:10.1016/j.ijplas.2004.04.014

1. Introduction

Microstructures of composite materials can be tailored to have desired overall properties. To this end, a quantitative relation between microstructural parameters and the overall property of composite materials must be established. Micromechanics accomplishes exactly this objective, and it is seen a rapid development in the past decades. For predicting the elastic property of composite materials, bounding methods, proposed by Hashin and Strikman (1963), and by Milton (1981) (see for example Torquato, 1991 for a review) can restrict the range of effective moduli in case of limited available microstructure information; approximate methods with particular microstructures such as Mori–Tanaka's model (Mori and Tanaka, 1973), double-inclusion model (Nemat-Nasser and Hori, 1993), self-consistent (Hershey, 1954; Kroner, 1958) or generalized self-consistent method (see for example Christensen and Lo, 1979) are also widely used for predicting the overall elastic property of composites. Their interconnections are recently discussed by Hu and Weng (2000a).

As concern as predicting the overall plasticity of composite materials, linearized methods such as secant moduli method and its modification (Tandon and Weng, 1988; Qiu and Weng, 1992; Hu, 1996); the variational method (Ponte Castañeda, 1991) and the unit cell model (Bao et al., 1991; Ji and Wang, 2003) are usually used for such purpose. These methods provide useful tools to bridge the macroscopic property of a composite with its microstructural parameters. However they all fail to predict the influence of particle's size on the overall plastic behavior for composite materials, since all the continuum formulations do not include inherently an intrinsic length scale. A considerable number of experiments reveals that composites do display different overall properties when their constituents take different absolute sizes, providing that all the other microscopic parameters remain unchanged. The finer the particle size, the harder the overall plastic behavior (Yang et al., 1990; Lloyd, 1994). For example, for the composites with a Al–4wt% Mg alloy matrix and 50% SiC particles, the size of SiC particles ranges from 13 to 165 μm, the macroscopic stress of the composites at 3.5% macroscopic deformation can vary from 575 to 375 MPa. Another well-known example is Hall–Petch relation for polycrystals between their yield stress and the grain size.

There are two different approaches to predict this size dependence, the first one is based on the viewpoint of material science, the geometrical necessary dislocations developed in a non-uniform stress field can influence the plastic behavior of materials (Fleck et al., 1994). This approach is more fundamental, and it has a clear physical picture. A recent advance of such theory is Taylor-based non-local plasticity theory (Gao and Huang, 2001), in which intrinsic material characteristic length is explained from Taylor's dislocation model. Recently, Hwang et al. (2004) developed the finite deformation version of this theory. Rashid and George (2004) introduced the coupling between the statistically stored and geometrically necessary dislocations into the formulation of the intrinsic length scale, experimental methods are also presented to determine this intrinsic length. For modeling the overall behavior of composites, Nan and Clarke (1996) used both continuum micromechanics and dislocation plasticity, in order to correlate better with experimental results, they additionally introduced particle cracking effect. Dai et al. (1999) included the geometrically necessary dislocation to explain the particle size dependence for metal matrix composites. The second approach is based on the viewpoint of continuum mechanics, this approach is more phenomenological. The basic idea is to consider the non-local response of a continuum, this leads to high order continuum theories. In this paper, we will follow the second approach. Recently, Shu and Barlow (2000) and Wei (2001) proposed a unit cell model, in which the matrix material is considered as a generalized continuum. The observed size dependence can be reproduced by these models through a finite element method after choosing carefully the intrinsic length parameters.

It is widely accepted that high order continuum theories with inherent length parameters can display some scale dependence. Generally speaking, there are four types of high order theories presented in the literature: non-local theory; gradient plasticity; Mindlin's generalized strain gradient theory (couple stress) and micropolar theory. In the first theory (Bazant and Pijaudier-Cabot, 1988), the response of a material at a point is determined not only by the state at that point but also by the deformation of its neighborhood. The second theory (Aifantis, 1984) adds gradients of state variables, in this approach, the yield function or plastic multiplier depends on second order spatial derivatives of the effective plastic strain, which introduces a length scale. The recent development of this theory is implicit gradient-enhanced plasticity (Roy et al., 2003), which involves the high order derivatives of non-local variables rather than local ones. The third one (Mindlin, 1963, 1964) introduces higher-order strain gradients in addition to original strain and stress measures, however this theory does not bring in new basic degrees of freedom, and this theory is often referred to as couple stress theory. In micropolar (Cosserat) theory (Cosserat, 1909; Toupin, 1962; Eringen, 1968), three micro-rotations are introduced in addition to the conventional three displacements at each material point, this leads to a non-symmetric stress (strain) and a high order couple stress (torsion). This theory was proposed as early as 1909, after a silence of about half a century, it has received great attention recently in exploring the scale effect presented in heterogeneous materials. Recently, Cheng and He (1995, 1997) derived an analytical solution for an inclusion problem in a micropolar medium, their work will be used in this paper to build the localization relation. In addition to elasticity, recently, Muhlhaus and Vardoulakis (1987), Vardoulakis (1989) and De Borst (1993) proposed a plasticity theory for a micropolar continuum (see more discussions by Lippmann, 1995). Fleck and Hutchinson (1997) also proposed a deformation and a flow version of plasticity for couple stress theory, a special case of micropolar theory. High order theories are valuable in simulation of localized phenomenon due to presence of microstructure in materials, as well as size dependence (De Borst, 1993; Kim and Oh, 2003). The application of high order theories hinged on the determination of the material's intrinsic length, there are some experimental efforts on this issue (e.g., Stolken and Evans, 1998; Wang et al., 2003).

Although there are many works devoted to micropolar theory, there is at present no corresponding analytical continuum micromechanical method in the framework of micropolar theory, which enables one to relate microstructural parameters to the overall plasticity for micropolar composites. Some efforts have been made in planar elasticity (Ostoja-Starzewski et al., 1999) or elastoplasticity (Forest et al., 2000), however these methods are basically numerical in nature. The objective of this paper is to propose an analytical micromechanical method in the framework of micropolar theory and to explain the observed particle size dependence for composite materials. The paper is arranged as follows: A brief introduction for micropolar elasticity and plasticity will be given in Section 2; A basic transition principle from micro to macroscopic scale for micropolar composites will be outlined in Section 3; in Section 4, a generalized Mori–Tanaka model for micropolar composites will be proposed to account for particle interactions; in Section 5, a secant moduli method will be proposed to predict the plastic behavior of composite materials; and finally in Section 6, numerical calculations will be performed to illustrate the predictive capability of the proposed model, the prediction will also be compared with experimental results in literature.

2. A brief review of micropolar theory

2.1. Micropolar elasticity

Micropolar theory assumes that not only forces but also moments can be transmitted across a surface element of a body, and in addition to the displacements, three extra degrees of freedom are introduced, characterizing the micro-rotation of each material point. The gradient of the rotation vector is defined as a torsion tensor, it is related by the constitutive relation to a couple stress tensor. Both the new introduced couple stress (torsion) and the classical stress (strain) are non-symmetric. A well-posed micropolar boundary value problem is properly established by the following three sets of governing equations (body forces and couples are neglected),

Kinematical relations:

$$\varepsilon_{ij} = u_{j,i} - e_{kij}\phi_k, \tag{1a}$$

$$k_{ij} = \phi_{j,i}. \tag{1b}$$

Balance equations:

$$\sigma_{ij,k} = 0, \tag{2a}$$

$$m_{ij,k} + e_{jik}\sigma_{ik} = 0. \tag{2b}$$

Constitutive equations

$$\sigma_{ji} = C_{jikl}\varepsilon_{kl} + B_{jikl}k_{kl}, \tag{3a}$$

$$m_{ji} = B_{jikl}\varepsilon_{kl} + D_{jikl}k_{kl}, \tag{3b}$$

with the following boundary conditions,

$$\sigma_{ij}n_i = p_j \quad m_{ij}n_i = z_j \quad \text{on } \Gamma^\sigma, \tag{4}$$

$$u_i = u_i^b \quad \phi_i = \phi_i^b \quad \text{on } \Gamma^u, \tag{5}$$

where σ_{ij} and m_{ij} are respectively stress and couple stress tensors, ε_{ij} and k_{ij} are the corresponding strain and torsion tensors; u_i and ϕ_i are displacement and micro-rotation vectors; B_{ijkl}, C_{ijkl} and D_{ijkl} are elastic constant tensors, p_j and z_j are surface force and moment vectors, n_i is exterior unit normal, e_{ijk} is the permutation tensor. Repeated indices imply a summation over 1, 2, 3. Subscripts preceded by a comma denote the derivatives with respect to the corresponding spatial coordinates.

For a centro-symmetric and isotropic micropolar continuum considered in this paper, $B_{jikl} = 0$, and

$$C_{jikl} = \left(\mu + \frac{\kappa}{2}\right)\delta_{jk}\delta_{il} + \left(\mu - \frac{\kappa}{2}\right)\delta_{ik}\delta_{jl} + \lambda\delta_{ij}\delta_{kl}, \tag{6a}$$

$$D_{jikl} = \gamma\delta_{jk}\delta_{il} + \beta\delta_{ik}\delta_{jl} + \alpha\delta_{ij}\delta_{kl}, \tag{6b}$$

where μ, λ are the classical Lame's constants, while κ, γ, β, α are new material constants introduced in micropolar theory, δ_{ij} is Kronecker delta.

If we denote by $\sigma'_{(ij)}$, $\sigma_{\langle ij\rangle}$, $\bar{\sigma}$ and $\varepsilon'_{(ij)}$, $\varepsilon_{\langle ij\rangle}$, $\bar{\varepsilon}$ respectively the deviatoric parts of symmetric stress and strain, anti-symmetric and hydrostatic parts of the stress and strain tensors, and similar notations for the couple-stress and torsion, the isotropic constitutive equations can be rewritten in the following form (Nowacki, 1986):

$$\sigma'_{(ij)} = 2\mu\varepsilon'_{(ij)} \quad \sigma_{\langle ij\rangle} = \kappa\varepsilon_{\langle ij\rangle} \quad \bar{\sigma} = 3K\bar{\varepsilon}, \tag{7a}$$

$$m'_{(ij)} = (\gamma + \beta)k'_{(ij)} \quad m_{\langle ij\rangle} = (\gamma - \beta)k_{\langle ij\rangle} \quad \bar{m} = 3L\bar{k}, \tag{7b}$$

where $K = \lambda + \frac{2}{3}\mu$, $L = \alpha + \frac{1}{3}(\gamma + \beta)$, K is the bulk modulus, L can be regarded as the corresponding measure for the torsion. The symbols in the subscript, () and $\langle\rangle$, denote the symmetric and anti-symmetric parts of a tensor, respectively. With such notation, the strain energy density for an isotropic micropolar continuum can be expressed as

$$U = \mu\varepsilon'_{(ij)}\varepsilon'_{(ij)} + \frac{\kappa}{2}\varepsilon_{\langle ij\rangle}\varepsilon_{\langle ij\rangle} + \frac{9}{2}K\bar{\varepsilon}^2 + \frac{\gamma+\beta}{2}k'_{(ij)}k'_{(ij)} + \frac{\gamma-\beta}{2}k_{\langle ij\rangle}k_{\langle ij\rangle} + \frac{9}{2}L\bar{k}^2. \tag{8}$$

We have two distinct sets of moduli: μ, λ, κ relate traditional stresses and strains, they have dimension of force per unit area; and γ, β, α relate high order couple-stresses and torsions, they have dimension of force. Due to the difference in the dimension between the two sets of moduli, these define some intrinsic characteristic lengths in a micropolar material. The characteristic length can be defined in different ways, in this paper, we consider:

$$l_1^2 = \frac{\gamma}{\mu} \quad l_2^2 = \frac{\beta}{\mu} \quad l_3^2 = \frac{\alpha}{\mu} \tag{9}$$

as three different intrinsic lengths.

2.2. Micropolar plasticity

The traditional J_2-flow theory can be extended to handle the plasticity for a micropolar material, a deformation version of micropolar plasticity will be presented in this paper. With the assumption that the hydrostatic parts of stress and couple-stress do not participate in plastic deformation, a new von Mises-like strain invariant for a micropolar material can be defined as (Lippmann, 1995)

$$\bar{\varepsilon}_{\mathrm{e}} = \sqrt{\frac{2}{3}\left\{\varepsilon'_{(ij)}\varepsilon'_{(ij)} + b_1\varepsilon_{\langle ij\rangle}\varepsilon_{\langle ij\rangle} + b_2\left[(\bar{l}_1^2 + \bar{l}_2^2)k'_{(ij)}k'_{(ij)} + \left(\bar{l}_1^2 - \bar{l}_2^2\right)k_{\langle ij\rangle}k_{\langle ij\rangle}\right]\right\}}. \tag{10}$$

In the absence of micropolar effect, Eq. (10) will reduce to the classical I_2 invariant. b_i and $\bar{l}_i$ are material constants, $\bar{l}_i$ can be considered as characteristic lengths at plastic state (Lippmann, 1995). The parameters b_i mean that only a portion of the anti-symmetric stress and couple stress participates in plastic hardening. For simplicity, we will consider in the following only two special cases, the case I: $b_1 = 0$, $b_2 = 1$; and the case II: $b_1 = 1$, $b_2 = 1$. The first case means that the anti-symmetric strain does not take part in plastic hardening, it will be shown in Section 5 that this case gives better correlation with experimental results. In the following we omit the symbol b_2 and let $b_1 = b$.

The strain energy density of a micropolar continuum is assumed to be

For case I

$$w = w_0(\bar{\varepsilon}_{\mathrm{e}}) + \frac{1}{2}\kappa\varepsilon_{\langle ij\rangle}\varepsilon_{\langle ij\rangle} + \frac{9}{2}K\bar{\varepsilon}^2 + \frac{9}{2}L\bar{k}^2. \tag{11}$$

For case II

$$w = w_0(\bar{\varepsilon}_{\mathrm{e}}) + \frac{9}{2}K\bar{\varepsilon}^2 + \frac{9}{2}L\bar{k}^2. \tag{12}$$

On the other hand, we have:

$$\delta w = \sigma'_{(ij)}\,\delta\varepsilon'_{(ij)} + \sigma_{\langle ij\rangle}\,\delta\varepsilon_{\langle ij\rangle} + m'_{(ij)}\,\delta k'_{(ij)} + m_{\langle ij\rangle}\,\delta k_{\langle ij\rangle} + 3\bar{\sigma}\,\delta\bar{\varepsilon} + 3\bar{m}\,\delta\bar{k}. \tag{13}$$

This gives for the stress (couple stress) and strain (torsion) relations ($b = 0$ for the type I and $b = 1$ for the type II):

$$\sigma'_{(ij)} = \frac{\partial w}{\partial\varepsilon'_{(ij)}} = \bar{\sigma}_{\mathrm{e}}\frac{\partial\bar{\varepsilon}_{\mathrm{e}}}{\partial\varepsilon'_{(ij)}} = \frac{2}{3}\,\frac{\bar{\sigma}_{\mathrm{e}}}{\bar{\varepsilon}_{\mathrm{e}}}\varepsilon'_{(ij)}, \tag{14a}$$

$$\sigma_{\langle ij\rangle} = b\frac{2}{3}\,\frac{\bar{\sigma}_{\mathrm{e}}}{\bar{\varepsilon}_{\mathrm{e}}}\varepsilon_{\langle ij\rangle} + (1-b)\kappa\varepsilon_{\langle ij\rangle}, \tag{14b}$$

$$m'_{(ij)} = (\bar{l}_1^2 + \bar{l}_2^2)\frac{2}{3}\,\frac{\bar{\sigma}_{\mathrm{e}}}{\bar{\varepsilon}_{\mathrm{e}}}k'_{(ij)}, \tag{14c}$$

$$m_{\langle ij\rangle} = \left(\bar{l}_1^2 - \bar{l}_2^2\right)\frac{2}{3}\,\frac{\bar{\sigma}_{\mathrm{e}}}{\bar{\varepsilon}_{\mathrm{e}}}k_{\langle ij\rangle}, \tag{14d}$$

$$\bar{\sigma} = 3K\bar{\varepsilon}, \tag{14e}$$

$$\bar{m} = 3L\bar{k}. \tag{14f}$$

Here we use the definition of $w_0(\bar{\varepsilon}_e)$, that is $\delta w_0(\bar{\varepsilon}_e) = \bar{\sigma}_e\,\delta\bar{\varepsilon}_e$, where $\bar{\sigma}_e$ is the corresponding equivalent stress. The traditional power type law for plastic hardening is generalized to a micropolar material with the above defined equivalent strain and stress, it can be written as

$$\bar{\sigma}_e = \sigma_y + H\bar{\varepsilon}_{ep}^n, \tag{15a}$$

$$\bar{\varepsilon}_e = \frac{\bar{\sigma}_e}{3\mu} + \left(\frac{\bar{\sigma}_e - \sigma_y}{H}\right)^{1/n}, \tag{15b}$$

where σ_y, H, n are initial yield stress and hardening parameters determined from a uniaxial tensile test, μ is the shear modulus.

With help of the defined hardening law and Eq. (14), the stress and couple stress can be completely determined provided that the strain and torsion are known. This deformation version of micropolar plasticity will be used to define the secant moduli of the matrix in Section 6. It is interesting to note that this deformation version of micropolar plasticity (the type I) is identical to a recent model proposed by Chen and Wang (2001) from a different perspective.

3. Micro–macro transition principle for a micropolar composite

Prediction of effective property for heterogeneous materials requires an appropriate definition of boundary conditions on a representative volume element (RVE) and the solution of this boundary value problem. The boundary condition must be specified to produce a well-posed structural problem, different boundary conditions are discussed by Forest et al. (1999). In this paper, we will examine the influence of particle size on the overall property of particulate composite materials. We are interested in the case where the particle size is small enough compared to the size of RVE, at the same time, the size of RVE is also small compared to the structural size. In this case, the macroscopic load gradient can be neglected on RVE (the size of RVE is small enough), this means the effective medium can be considered as Cauchy one. For classical micromechanics, the microstructure of the matrix material is ignored, and the matrix material is idealized to be a Cauchy material model, this in fact implies that the size of the particle must be much larger than the intrinsic length of the matrix (for example the size of the grain for polycrystals). However, for metal matrix composites, the intrinsic length of the matrix is comparable to the size of the particle, in this situation, the non-local response of the matrix material must be taken into account to some extend in a proper continue modeling. To this end, we consider the local constituents to be micropolar materials. The following traditional boundary conditions are prescribed on the boundary of RVE:

$$u_j = E_{(ij)}x_i \quad \phi_j = 0, \tag{16}$$

where $E_{(ij)}$ is symmetric and constant over RVE. For any statically balanced local stresses (σ_{ij}, m_{ij}) and geometrically compatible local strain fields $(\varepsilon_{ij}, k_{ij})$, the volume average of the internal energy over RVE is

$$\begin{aligned}\langle \sigma_{ij}\varepsilon_{ij} + m_{ij}k_{ij}\rangle &= \langle \sigma_{ij}(u_{j,i} - e_{kij}\varphi_k)\rangle + \langle m_{ij}\varphi_{j,i}\rangle \\ &= \frac{1}{V}\int_{\partial\mathrm{RVE}} \sigma_{ij}u_j n_i\,\mathrm{d}S + \frac{1}{V}\int_{\partial\mathrm{RVE}} m_{ij}\phi_j n_i\,\mathrm{d}S \\ &= E_{(mj)}\frac{1}{V}\int_{\partial\mathrm{RVE}} \sigma_{ij}x_m n_i\,\mathrm{d}S = E_{(mj)}\langle\sigma_{mj}\rangle = E_{(mj)}\langle\sigma_{(mj)}\rangle.\end{aligned} \tag{17}$$

On the other hand:

$$\langle\varepsilon_{(ij)}\rangle = \langle(\varepsilon_{ij} + \varepsilon_{ji})/2\rangle = \langle(u_{i,j} + u_{j,i})/2\rangle = E_{(ij)}, \tag{18}$$

where $\langle\cdot\rangle$ means the volume average of said quantity over RVE.

If a stress boundary condition is applied on the boundary of RVE

$$\sigma_{ij}n_i = \Sigma_{(ij)}n_i, \quad m_{ij}n_i = 0, \tag{19}$$

where n_i is outer normal of the boundary of RVE.

The energy equivalence (Eq. (17)) now can be written as

$$\langle \sigma_{ij}\varepsilon_{ij} + m_{ij}k_{ij}\rangle = \Sigma_{(ij)}\langle\varepsilon_{(ij)}\rangle \tag{20}$$

and

$$\langle\sigma_{(ij)}\rangle = \Sigma_{(ij)}. \tag{21}$$

So we can define the classical effective modulus tensor $\bar{\boldsymbol{C}}$ or effective compliance tensor $\bar{\boldsymbol{S}}$ by

$$\langle\boldsymbol{\sigma} : \boldsymbol{\varepsilon} + \boldsymbol{m} : \boldsymbol{k}\rangle = \boldsymbol{E}^{\mathrm{s}} : \bar{\boldsymbol{C}} : \boldsymbol{E}^{\mathrm{s}} = \Sigma^{\mathrm{s}} : \bar{\boldsymbol{S}} : \boldsymbol{\Sigma}^{\mathrm{s}}, \tag{22}$$

where the superscript ‘s’ means a symmetric tensor.

4. Generalized Mori–Tanaka’s model for micropolar composites

4.1. Solution of a spherical inclusion problem in an infinite micropolar matrix

In a classical Cauchy medium, Eshelby (1957) obtained the solution for an infinite medium in which a spheroidal region is subjected to a uniform eigen-strain, the resulting stress is uniform in this region. For a micropolar medium, Cheng and He (1995, 1997) formulated the same problem for a spherical and a long cylindrical inclusions respectively. The problem that they solved can be stated as follows: in an infinite uniform micropolar medium, there is an isolated spherical (circular) region with a prescribed uniform eigen-strain $\boldsymbol{\varepsilon}^{+}$ and eigen-torsion $\boldsymbol{k}^{+}$, the solution for the resulted strain and torsion in the whole medium can be expressed by (Cheng and He, 1995, 1997):

$$\boldsymbol{\varepsilon} = \boldsymbol{K}(\boldsymbol{x}) : \boldsymbol{\varepsilon}^{+} + \boldsymbol{L}(\boldsymbol{x}) : \boldsymbol{k}^{+}, \tag{23a}$$

$$\boldsymbol{k} = \hat{\boldsymbol{K}}(\boldsymbol{x}) : \boldsymbol{\varepsilon}^{+} + \hat{\boldsymbol{L}}(\boldsymbol{x}) : \boldsymbol{k}^{+}, \tag{23b}$$

where $\boldsymbol{K}$, $\hat{\boldsymbol{K}}$, $\boldsymbol{L}$, $\hat{\boldsymbol{L}}$ are Eshelby-like-tensors, and they are given by Cheng and He (1995, 1997), $\boldsymbol{x}$ is a position vector.

It must be pointed that, unlike the Eshelby's solution, even for a spherical inclusion, the resulted strain and torsion in the inclusion domain are not uniform. This makes the equivalent inclusion method (Mura, 1982) widely used for a Cauchy composite difficult to be applied exactly for a micropolar composite. For example, a spherical inhomogeneity in an infinite micropolar matrix, its effect on the stress and couple stress distribution can not be simulated by the same form inclusion with a *uniform* eigen-strain and eigen-torsion, rather with some *non-uniform* eigen-strain and eigen-torsion.

With this limitation in mind, we postulate that the equivalent inclusion method could be applied in an average sense for a micropolar composite, since in an elastic case, only the average stress and couple stress in an inhomogeneity will be used to determine the effective property. For a spherical particle with moduli $\boldsymbol{C}_1, \boldsymbol{D}_1$, and it is embedded into a micropolar matrix $\boldsymbol{C}_0, \boldsymbol{D}_0$ under a remote loading $\boldsymbol{E}_0, \boldsymbol{K}_0$, we have according to the average equivalent inclusion method

$$\boldsymbol{C}_1 : (\boldsymbol{E}_0 + \langle \boldsymbol{\varepsilon} \rangle_{\mathrm{I}}) = \boldsymbol{C}_0 : (\boldsymbol{E}_0 + \langle \boldsymbol{\varepsilon} \rangle_{\mathrm{I}} - \langle \boldsymbol{\varepsilon}^{+} \rangle_{\mathrm{I}}), \tag{24a}$$

$$\boldsymbol{D}_1 : (\boldsymbol{K}_0 + \langle \boldsymbol{k} \rangle_{\mathrm{I}}) = \boldsymbol{D}_0 : (\boldsymbol{K}_0 + \langle \boldsymbol{k} \rangle_{\mathrm{I}} - \langle \boldsymbol{k}^{+} \rangle_{\mathrm{I}}), \tag{24b}$$

where $\langle \cdot \rangle_{\mathrm{I}}$ means the volume average over the inclusion domain. Averaging Eq. (23a) and (23b) over the particle, it has

$$\langle \boldsymbol{\varepsilon} \rangle_{\mathrm{I}} = \langle \boldsymbol{K} \rangle_{\mathrm{I}} : \langle \boldsymbol{\varepsilon}^{+} \rangle_{\mathrm{I}} + \langle \boldsymbol{L} \rangle_{\mathrm{I}} : \langle \boldsymbol{k}^{+} \rangle_{\mathrm{I}}, \tag{25a}$$

$$\langle \boldsymbol{k} \rangle_{\mathrm{I}} = \langle \hat{\boldsymbol{K}} \rangle_{\mathrm{I}} : \langle \boldsymbol{\varepsilon}^{+} \rangle_{\mathrm{I}} + \langle \hat{\boldsymbol{L}} \rangle_{\mathrm{I}} : \langle \boldsymbol{k}^{+} \rangle_{\mathrm{I}}. \tag{25b}$$

In a strict sense, such a simplification does not take all the equivalent information into account, we would like to call it a *weak-form* equivalence. In a separate work (Xun et al., 2004), we determine exactly the stress and couple stress distributions for an infinite micropolar medium with a cylindrical fiber (plane strain condition) under a remote tensile loading, the average stress and couple stress in the fiber are compared with those obtained by the average equivalent inclusion method described above, and it is found that they agree very well with each other.

In order to proceed, the volume averages over the particle for the Eshelby-like tensors $\boldsymbol{K}$, $\hat{\boldsymbol{K}}$, $\boldsymbol{L}$, $\hat{\boldsymbol{L}}$ have to be estimated. The detailed results are given in Appendix A, it is found after averaging that

$$\langle \boldsymbol{L} \rangle_{\mathrm{I}} = 0 \quad \text{and} \quad \langle \hat{\boldsymbol{K}} \rangle_{\mathrm{I}} = 0. \tag{26}$$

This implies, in an average sense, that an eigen-strain produces on average only strain, and an eigen-torsion leads only to an average torsion deformation, they are uncoupled. These results agree with those obtained by Yang (2000), who took Taylor series expansion of the Eshelby-like-tensors inside a spherical inclusion, and then performed the average.

4.2. Effective elastic moduli estimated by generalized Mori–Tanaka method

With the solution of a single inclusion in a micropolar matrix, now we are ready to estimate the effective modulus tensor defined by Eq. (22). To do this, we have to write Eq. (25) in form of symmetric and anti-symmetric parts separately. For a spherical inclusion in an isotropic micropolar matrix, the Eshelby-like tensor is an isotropic fourth order tensor, i.e., it always has the following form:

$$K_{ijkl} = T_1\delta_{ij}\delta_{kl} + (T_2 + T_3)\delta_{ik}\delta_{jl} + (T_2 - T_3)\delta_{il}\delta_{jk}.$$

So the volume average of the micropolar Eshelby-like relations can be decomposed into symmetric and anti-symmetric parts, respectively:

$$\langle \varepsilon_{(ij)} \rangle_{\mathrm{I}} = \langle K^{\mathrm{S}}_{ijkl} \rangle_{\mathrm{I}} : \langle \varepsilon^{+}_{(kl)} \rangle_{\mathrm{I}}, \tag{27a}$$

$$\langle \varepsilon_{\langle ij\rangle} \rangle_{\mathrm{I}} = \langle K^{\mathrm{A}}_{ijkl} \rangle_{\mathrm{I}} : \langle \varepsilon^{+}_{\langle kl\rangle} \rangle_{\mathrm{I}}, \tag{27b}$$

where $K^{\mathrm{S}}_{ijkl} = (K_{ijkl} + K_{ijlk})/2$, $K^{\mathrm{A}}_{ijkl} = (K_{ijkl} - K_{ijlk})/2$.

With help of Eq. (27), and writing Eq. (24) into a symmetric and an anti-symmetric parts, the estimate of the effective moduli $\bar{\boldsymbol{C}}$ for a micropolar composite can follow exactly the same procedure as for a Cauchy composite, this gives (Hu and Weng, 2000a,b)

$$\bar{\mu} = \mu_0\left\{1 + \frac{f}{2(1-f)\langle K^{\mathrm{S}}_{1212}\rangle_{\mathrm{I}} + [\mu_0/(\mu_1 - \mu_0)]}\right\}, \tag{28a}$$

$$\bar{K} = K_0\left\{1 + \frac{f}{[(1-f)\langle K^{\mathrm{S}}_{iijj}\rangle_{\mathrm{I}}/3] + [K_0/(K_1 - K_0)]}\right\}, \tag{28b}$$

where f is the volume fraction of particles, $\bar{K}$, K_1 and K_0 are respectively the bulk moduli of the composite, particle and matrix, and $\bar{\mu}$, μ_1 andμ_0 are the corresponding shear moduli. The scale information is included in the Eshelby-like-tensor implicitly through $\langle \boldsymbol{K}^{\mathrm{S}}(\boldsymbol{x})\rangle_{\mathrm{I}}$, the analytical expressions of $\langle K^{\mathrm{S}}_{1212}\rangle_{\mathrm{I}}$ and $\langle K^{\mathrm{S}}_{iijj}\rangle_{\mathrm{I}}$ are given in Appendix A. When particle size is large or the micropolar effect is neglected, $\langle \boldsymbol{K}^{\mathrm{S}}(\boldsymbol{x})\rangle_{\mathrm{I}}$ is reduced to the classical Eshelby tensor, and the estimate for the effective bulk and shear moduli of the composite is reduced to the classical one.

5. Secant moduli method for overall plasticity of micropolar composites

5.1. Second order moment of strain and torsion of the matrix

For composite materials, distributions of stress and strain in the matrix material are not uniform and they are usually very complicated, since they depend intimately on the form and distribution of reinforced phases and also on external loading level. In this paper for simplification of notation, we discuss the case where only the matrix material can have a plastic deformation, and the particle remains elastic. When the effective stress (strain) in the matrix arrives at in an average sense the yielding point, then the composite starts to yield. It should be mentioned that the proposed method can be applied for any multi-phase composite and for more than one plastic phase. In order to estimate the non-linear behavior of a composite, the equivalent strain defined by Eq. (10) under any given load has to be evaluated. In this paper, an analytical method will be proposed to calculate this quantity by estimating the second moment soft strain and torsion, just as for a Cauchy composite (Qiu and Weng, 1992; Hu, 1996).

Under an affine displacement boundary condition (prescribed by Eq. (16)), let local stiffness tensors $\delta\boldsymbol{C}(\boldsymbol{x})$ and $\delta\boldsymbol{D}(\boldsymbol{x})$ have independent variations, this will lead to variations of the local strain and torsion fields $\delta\boldsymbol{\varepsilon}(\boldsymbol{x})$, $\delta\boldsymbol{k}(\boldsymbol{x})$, which in turn leads to a variation of the composite stiffness tensor $\delta\bar{\boldsymbol{C}}$, we have according to Eq. (22)

$$\boldsymbol{E}^{\mathrm{s}} : \delta\bar{\boldsymbol{C}} : \boldsymbol{E}^{\mathrm{s}} = \langle \boldsymbol{\varepsilon} : \delta\boldsymbol{C} : \boldsymbol{\varepsilon} + \boldsymbol{k} : \delta\boldsymbol{D} : \boldsymbol{k} \rangle + 2\langle \boldsymbol{\varepsilon} : \boldsymbol{C} : \delta\boldsymbol{\varepsilon} + \boldsymbol{k} : \boldsymbol{D} : \delta\boldsymbol{k} \rangle. \tag{29}$$

Since $\boldsymbol{\varepsilon} : \boldsymbol{C}$, $\boldsymbol{k} : \boldsymbol{D}$ are the statically balanced stress and couple-stress, and $\delta\boldsymbol{\varepsilon}(\boldsymbol{x})$, $\delta\boldsymbol{k}(\boldsymbol{x})$ are the geometrically compatible strain and torsion fields corresponding to zero boundary displacements. With help of Eq. (17), and $\delta\boldsymbol{u} = \delta\boldsymbol{E}^{\mathrm{s}} \cdot \boldsymbol{x} = 0$, $\delta\boldsymbol{\phi} = 0$ on the boundary of RVE, the second term of the right hand side of Eq. (29) becomes

$$\langle \boldsymbol{\varepsilon} : \boldsymbol{C} : \delta\boldsymbol{\varepsilon} + \boldsymbol{k} : \boldsymbol{D} : \delta\boldsymbol{k} \rangle = 0. \tag{30}$$

Therefore we arrive at

$$\boldsymbol{E}^{\mathrm{s}} : \delta\bar{\boldsymbol{C}} : \boldsymbol{E}^{\mathrm{s}} = \langle \boldsymbol{\varepsilon} : \delta\boldsymbol{C} : \boldsymbol{\varepsilon} + \boldsymbol{k} : \delta\boldsymbol{D} : \boldsymbol{k} \rangle. \tag{31}$$

We rewrite Eq. (22) as

$$U = \frac{1}{2}\langle \boldsymbol{\varepsilon} : \boldsymbol{C} : \boldsymbol{\varepsilon} + \boldsymbol{k} : \boldsymbol{D} : \boldsymbol{k} \rangle = \frac{1}{2}\boldsymbol{E}^{\mathrm{s}} : \bar{\boldsymbol{C}} : \boldsymbol{E}^{\mathrm{s}}. \tag{32}$$

For an isotropic elastic micropolar material, the left-hand side of the above equation can be written as (see Eq. (8))

$$\begin{aligned} U = \Big\langle & \mu\varepsilon'_{(ij)}\varepsilon'_{(ij)} + \frac{\kappa}{2}\varepsilon_{\langle ij\rangle}\varepsilon_{\langle ij\rangle} + \frac{1}{2}K\varepsilon_{kk}\varepsilon_{nn} + \frac{\gamma+\beta}{2}k'_{(ij)}k'_{(ij)} \\ & + \frac{\gamma-\beta}{2}k_{\langle ij\rangle}k_{\langle ij\rangle} + \frac{1}{2}Lk_{kk}k_{nn} \Big\rangle. \end{aligned} \tag{33}$$

Let the matrix elastic constants μ_0, κ_0, γ_0 and β_0 undergo independent variations, $\delta\mu_0$, $\delta\kappa_0$, $\delta\gamma_0$ and $\delta\beta_0$ respectively, the other constants remain unchanged, the following exact equations can be derived:

$$(1-f)\langle \varepsilon'_{(ij)}\varepsilon'_{(ij)} \rangle_0 \delta(2\mu_0) = \boldsymbol{E}^{\mathrm{s}} : \delta\bar{\boldsymbol{C}} : \boldsymbol{E}^{\mathrm{s}}, \tag{34a}$$

$$(1-f)\langle \varepsilon_{\langle ij\rangle}\varepsilon_{\langle ij\rangle} \rangle_0 \delta(\kappa_0) = \boldsymbol{E}^{\mathrm{s}} : \delta\bar{\boldsymbol{C}} : \boldsymbol{E}^{\mathrm{s}}, \tag{34b}$$

$$(1-f)\langle k'_{(ij)}k'_{(ij)} + k_{\langle ij\rangle}k_{\langle ij\rangle}\rangle_0 \delta(\gamma_0) = \boldsymbol{E}^{\mathrm{s}} : \delta\bar{\boldsymbol{C}} : \boldsymbol{E}^{\mathrm{s}}, \tag{34c}$$

$$(1-f)\langle k'_{(ij)}k'_{(ij)} - k_{\langle ij\rangle}k_{\langle ij\rangle}\rangle_0 \delta(\beta_0) = \boldsymbol{E}^{\mathrm{s}} : \delta\bar{\boldsymbol{C}} : \boldsymbol{E}^{\mathrm{s}}, \tag{34d}$$

where $\langle\cdot\rangle_0$ means the volume average of the said quantity over the matrix domain.

Finally the average equivalent strain of the matrix defined by Eq. (10) can be evaluated for any applied strain by the following equation:

$$\langle \varepsilon_{\mathrm{eff}} \rangle_0 = \sqrt{\frac{1}{3(1-f)} \boldsymbol{E}^s : \boldsymbol{S} : \boldsymbol{E}^s}, \tag{35}$$

where

$$S = \frac{\partial \bar{\boldsymbol{C}}}{\partial \mu_0} + 2b\frac{\partial \bar{\boldsymbol{C}}}{\partial \kappa_0} + 2\bar{l}_1^2 \frac{\partial \bar{\boldsymbol{C}}}{\partial \gamma_0} + 2\bar{l}_2^2 \frac{\partial \bar{\boldsymbol{C}}}{\partial \beta_0}. \tag{36}$$

The effective stiffness tensor $\bar{\boldsymbol{C}}$ can be evaluated by the method proposed in Section 4, so the average effective strain of the matrix can then be computed, and the non-linear behavior of the composite can be estimated with help of a secant moduli method defined in the following.

5.2. Secant moduli of the matrix and composite

With the deformation theory presented in Section 2.2 for a micropolar plasticity and a power type law for plastic hardening, when the matrix undergoes a plastic deformation, the secant moduli of the matrix can be defined as (here we set $\bar{l}_1^2 = \bar{l}_2^2 = l_{\mathrm{p}}^2$ for simplification):

$$\mu^{\mathrm{s}} = \frac{1}{(1/\mu_0) + 3[(\bar{\sigma}_{\mathrm{e}} - \sigma_{\mathrm{y}})/H]^{1/n}/\bar{\sigma}_{\mathrm{e}}}, \tag{37a}$$

$$\kappa_0^{\mathrm{s}} = 2b\mu_0^{\mathrm{s}} + (1-b)\kappa_0, \tag{37b}$$

$$\gamma_0^{\mathrm{s}} = \beta_0^{\mathrm{s}} = 2l_{\mathrm{p}}^2\mu_0^{\mathrm{s}}, \tag{37c}$$

$$K_0^{\mathrm{s}} = K_0, \tag{37d}$$

$$L_0^{\mathrm{s}} = L_0, \tag{37e}$$

where $b = 0$ and $b = 1$ correspond respectively to the deformation theories of the type I and II described in Section 2.2. It is seen that for the deformation theory of the type II, there is a jump for the modulus κ_0 when a material passes from elasticity into plasticity.

With the above-defined secant moduli of the matrix in micropolar plasticity, we can follow exactly the same idea of the secant moduli method developed for a Cauchy composite (see for example Tandon and Weng, 1988; Qiu and Weng, 1992; Hu, 1996). This method can be explained by the following procedure: for any given macroscopic load $\boldsymbol{E}^{\mathrm{s}}$, at which the matrix has undergone a plastic deformation, for a given average effective stress of the matrix $\bar{\sigma}_{\mathrm{e}} > \sigma_{\mathrm{y}}$, the secant moduli of the matrix can be evaluated by Eq. (37), the stiffness tensors $\bar{\boldsymbol{C}}_{\mathrm{L}}$ of a linearized elastic composite with an elastic matrix having the secant moduli of the actual matrix can then be determined from Eq. (28), and these moduli are interpreted as the secant moduli of the actual composite. By repeating $\bar{\sigma}_{\mathrm{e}}$, the non-linear relation between the stress and strain of the composite material can be established. For a Cauchy composite, the prescribed secant moduli method corresponds exactly to the variational method proposed by Ponte Castañeda (1991) (see for example Suquet, 1995; Hu, 1996). It is also shown that the proposed method has also a similar variational structure (Hu et al., in press).

6. Numerical applications

In this section, the proposed micromechanical method will be illustrated by some numerical examples. In the following, we assumed the elastic characteristic lengths $l_1 = l_2 = l_3 = l_{\mathrm{m}}$ for simplification. A metal matrix composite (SiC/Al) is taken to be the sample material. The material constants of the constituents are listed in Table 1. We idealize the matrix as a micropolar material, and the SiC particles are considered as a Cauchy material, that is, the corresponding micropolar moduli are set to be zero.

Fig. 1 shows the predicted effective stress and strain curves of the composite with the deformation theory of the type I for different particle sizes, the volume fraction of the particles is set to be $f = 0.15$, and the plastic material characteristic length $l_{\mathrm{p}} = 2l_{\mathrm{m}}$. The particle radii are chosen to be 1, 5, 10 and 100 times the matrix characteristic length l_{m}. The stress is normalized by the matrix initial yield stress, and at the same time the classical micromechanical prediction and the matrix stress and strain curve are also included for comparison. When particle diameter approaches to the matrix's characteristic length l_{m}, the predicted effective stress and strain curves of the composite differs significantly from those predicted by classical micromechanical methods. When $a = l_{\mathrm{m}}$, the predicted stress can reach almost twice the initial yield

Table 1
Material constants of 6061 Al and SiC used in calculations

	6061 Al	SiC
E (GPa)	68.3	490
ν	0.33	0.17
μ (GPa)	26	209
λ (GPa)	50	108
κ (GPa)	26	
l_{m} (μm)	10	
σ_{y} (MPa)	250	
h (MPa)	173	
n	0.455	

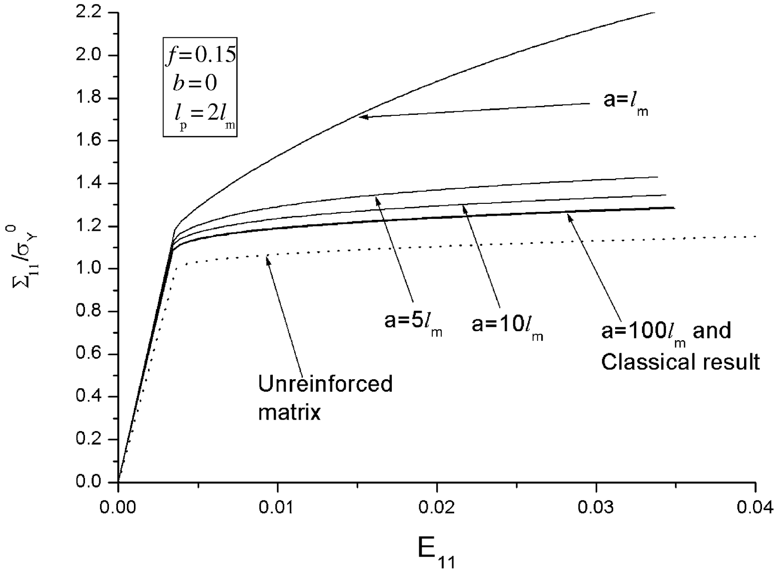

Fig. 1. Particle size effect predicted by the proposed micromechanical model with deformation theory of type I.

stress at 3.5% macroscopic strain; on the other hand when $a = 100l_{\mathrm{m}}$, the classical and micropolar predictions are almost the same.

Fig. 2 shows the predicted stress and strain curves by the deformation theory of the type I and II respectively, the volume fraction of particles is 30%, and the plastic material characteristic length $l_{\mathrm{p}} = 2l_{\mathrm{m}}$. It is also seen that the deformation theory of the type I for the matrix material gives more pronounced scale dependence than that predicted by the deformation theory of the type II. It is recalled that the anti-symmetric part of strain is not active in the plastic hardening for the deformation theory of the type I. This result means that the anti-symmetric shear modulus is an important parameter in micropolar theory, its elastic value and plastic evolution rule will definitely affect the predicted results, as discussed by Koiter (1964), and recently confirmed by Chen and Wang (2001). So in the following computation only the deformation theory of the type I is considered.

Fig. 3 shows the influence of the plastic characteristic length l_{p} on the prediction, two values of the matrix plastic characteristic lengths $l_{\mathrm{p}} = 2l_{\mathrm{m}}$ and $3l_{\mathrm{m}}$ are examined respectively. It is found that as l_{p} increases, more pronounced scale dependences are predicted.

The computed results for the composites with 30% rigid inclusions or voids are shown in Fig. 4, the matrix property remains the same for the both cases. It shows that the harder the particle, the more pronounced the particle size dependence. For voided materials, the predicted void's size dependence in tension can be neglected.

Finally we have compared the predictive capability of the proposed model with experimental results in literature. Two experimental results are taken for

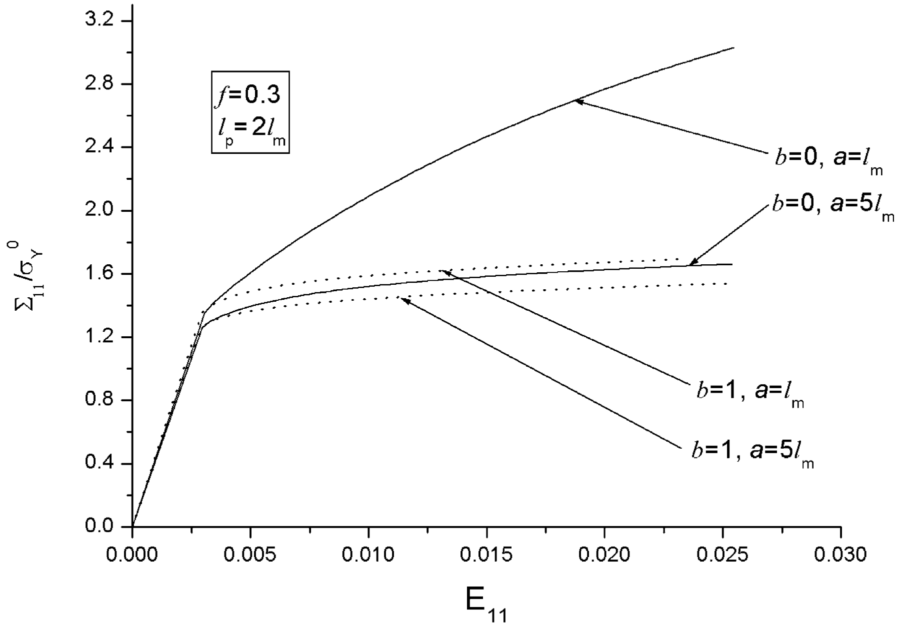

Fig. 2. Comparison of predicted effective stress and strain curves by deformation theories of the type I and II for different particle sizes.

comparison, the first one is a composite with the Al356 (T4) matrix and 15% SiC particles, analyzed by Lloyd (1994); the second is a composite with Al–4wt% Mg alloy as the matrix and 50% SiC particles, presented by Yang et al. (1990). In the micromechanical modeling of both composites, the deformation theory of the type I ($b = 0$) and plastic characteristic length $l_p = 2l_m$ are used. The material constants used in the computation are listed in Table 2. The elastic characteristic length $l_m = 0.75$ μm is taken for the first composite, and $l_m = 2$ μm for the second, these values can give a better fit to the experimental results, which will be shown in the following. We recall that at present there is almost no available experiment for determining these micropolar material constants. Nevertheless, these length scales must be in a reasonable range, they ought to be on the same magnitude order with those obtained from material science approaches. For typical metals the length scales of local non-uniform deformation is indeed on the order of micrometers (Fleck et al., 1994; Gao et al., 1999). Our choice of l_m conforms to this result.

Fig. 5 shows the comparison results for the first composite, the particle diameters are 7.5 and 16 μm respectively as determined by the experiment (Lloyd, 1994), an excellent correlation is found between the modeling and the experiment for a large range of macroscopic strain.

Fig. 6 illustrates the comparison results for the second composite, in the experiment, the composites with two very different particle sizes, 13 and 165 μm are examined. The volume fraction of the particles is 50% for the both cases. The

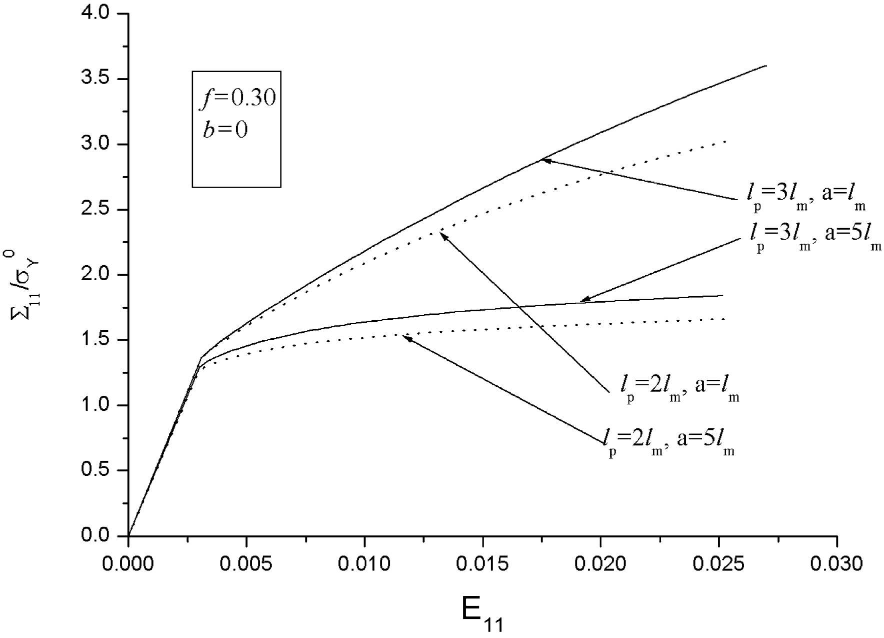

Fig. 3. Predicted effective stress and strain curve of composites for different plastic characteristic lengths l_p of the matrix and different particle sizes.

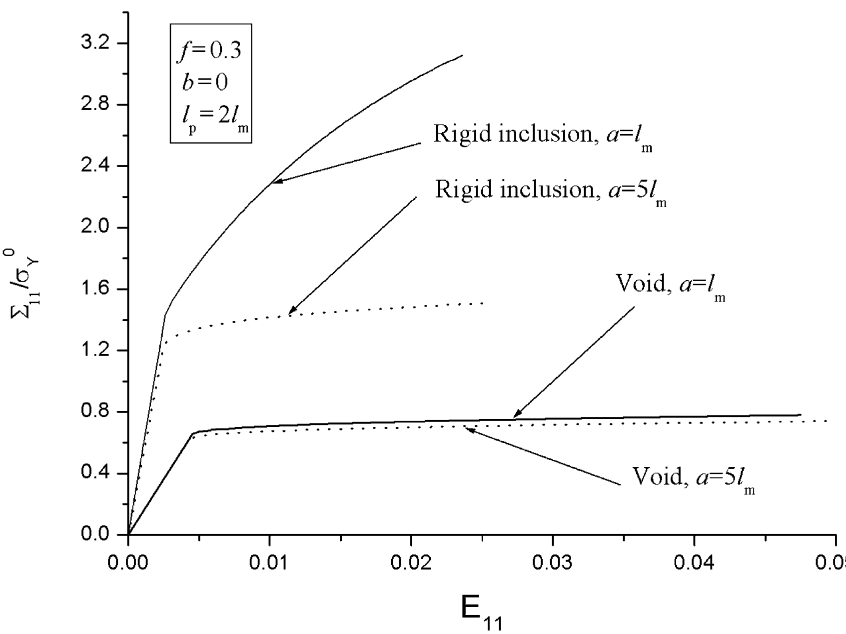

Fig. 4. Predicted effective stress and strain curves of composites for different elastic properties and sizes of particles.

Table 2
Material constants of each constituent used in comparison

	Al356(T4)	Al–4wt% Mg	SiC
E (GPa)	70	70	490
ν	0.33	0.33	0.17
μ (GPa)	26.3	26.3	209
λ (GPa)	51.1	51.1	108
κ (GPa)	26.3	26.3	
σ_y (MPa)	86	124.2	
h (MPa)	414.7	414.7	
n	0.365	0.365	

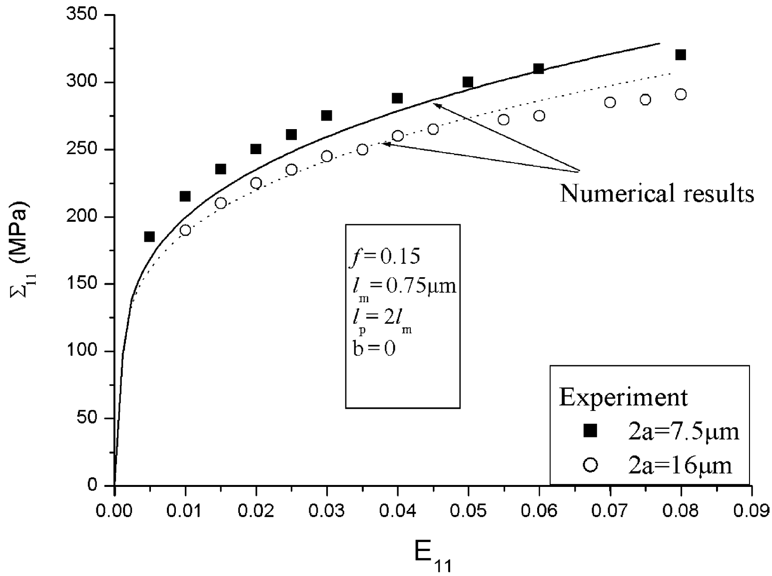

Fig. 5. Comparison between the modeling and Lloyd's (1994) experimental results for the effective stress and strain relations.

composites exhibit a strong particle size dependence due to a relatively high particle volume fraction. As for the first composite, the modeling results agree quite well with the experimental results performed by Yang et al. (1990).

Through the comparison with the experiment, we find that the proposed micromechanical model can indeed capture the particle size dependence of plasticity for composite materials. It must be mentioned that the residual stress is not considered in the modeling and the particle is considered to be spherical, the influence of these two factors will be considered in our further works. Finally, It must be emphasized that the proposed secant moduli method is independent of the way for evaluating the

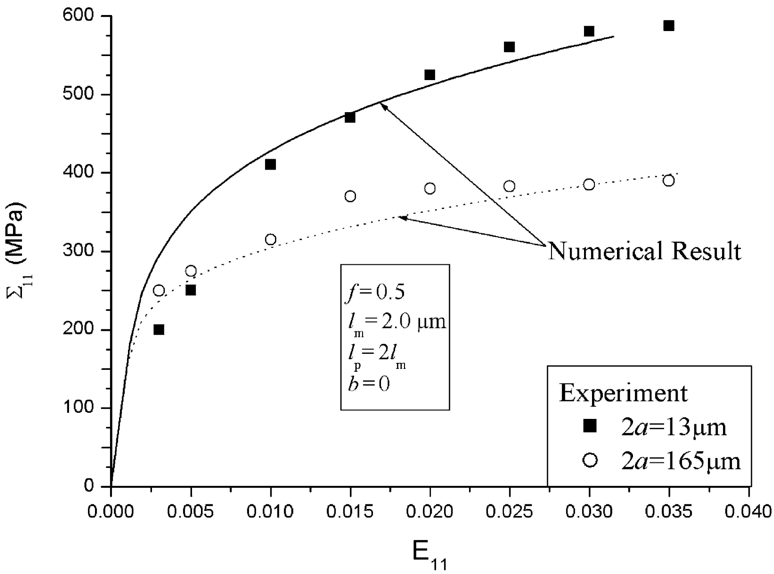

Fig. 6. Comparison between the modeling and Yang et al. (1990) experimental results for the effective stress and strain relations.

elastic stiffness of composites, so other elastic micromechanical models can also be considered with the proposed method.

7. Conclusions

An analytical micromechanical method in a framework of micropolar theory is proposed to investigate the size dependence on the overall plasticity of composite materials. The method is based on the average equivalent inclusion method for a micropolar material, and Mori–Tanaka's method is extended to a micropolar composite to evaluate the effective elastic stiffness tensor. A secant moduli method based on the second moment of strain and torsion of the matrix is then proposed to examine the non-linear behavior of composites. The proposed micromechanical method can be considered as a natural extension of the widely used secant moduli method from a Cauchy composite to a micropolar composite.

The influence of matrix characteristic lengths, size and property of the particle on overall plastic behavior of composites are examined. The size dependence is more pronounced for the case where the particle size approaches to the matrix characteristic length, and where the stiffness of the particle is large. When the particle size is much larger than the matrix characteristic length, the prediction based on micropolar theory is reduced to the classical results. The computed results also show that for voided materials the size effect of the void is not significant in case of a tensile loading. The prediction based on the proposed method agrees well with experimental results in literature.

Acknowledgements

This work is supported by National Nature Science Foundation of China through Grant 10325210.

Appendix A

According to Cheng and He (1995), the four Eshelby-like tensors $\boldsymbol{K}$, $\hat{\boldsymbol{K}}$, $\boldsymbol{L}$, $\hat{\boldsymbol{L}}$ are evaluated by the following three primitive potential integrals and their derivatives with respect to spatial coordinates,

$$\psi(\boldsymbol{x}) = \frac{1}{4\pi}\int_\Omega x\,\mathrm{d}\boldsymbol{x}', \quad \phi(\boldsymbol{x}) = \frac{1}{4\pi}\int_\Omega \frac{1}{x}\,\mathrm{d}\boldsymbol{x}', \quad M(\boldsymbol{x},k) = \frac{1}{4\pi}\int_\Omega \frac{\mathrm{e}^{-x/k}}{x}\,\mathrm{d}\boldsymbol{x}', \tag{A.1}$$

where $x = |\boldsymbol{x}|$ and Ω represents the inclusion domain, note that $\phi(\boldsymbol{x})$ has a different meaning as in the main text.

Therefore to average Eshelby-like tensors over the inclusion domain, we just need to average Eq. (A.1) and their derivatives. The first two integrals are identical to the classical inclusion theory (Cauchy theory) as discussed by Mura (1982), and the last one is new due to the presence of micropolar effect. Since $\psi(\boldsymbol{x})$ and $\phi(\boldsymbol{x})$ are just simple polynomial for a spherical inclusion, any manipulation of them is easy, we consider in the following only the average of $M(\boldsymbol{x},k)$. For an interior point of a spherical inclusion, $M(\boldsymbol{x},k)$ and its necessary derivatives are

$$M(\boldsymbol{x},k) = k^2 + \Phi(k) j_0\left(\frac{\mathrm{i}x}{k}\right), \tag{A.2a}$$

$$M_{,i}(\boldsymbol{x},k) = \Phi(k)\frac{x_i}{\mathrm{i}kx} j_1\left(\frac{\mathrm{i}x}{k}\right), \tag{A.2b}$$

$$M_{,ij}(\boldsymbol{x},k) = \Phi(k)\left[\frac{\delta_{ij}}{\mathrm{i}kx} j_1\left(\frac{\mathrm{i}x}{k}\right) + \frac{x_i x_j}{(\mathrm{i}kx)^2} j_2\left(\frac{\mathrm{i}x}{k}\right)\right], \tag{A.2c}$$

$$\begin{aligned} M_{,ijm}(\boldsymbol{x},k) = \Phi(k)\Bigg[&\frac{x_i\delta_{jm} + x_j\delta_{im} + x_m\delta_{ij}}{(\mathrm{i}kx)^2} j_2\left(\frac{\mathrm{i}x}{k}\right) \\ &+ \frac{x_i\delta_{jm} + x_j\delta_{im} + x_m\delta_{ij}}{(\mathrm{i}kx)^3} j_3\left(\frac{\mathrm{i}x}{k}\right)\Bigg], \end{aligned} \tag{A.2d}$$

$$\begin{aligned} M_{,ijmn}(\boldsymbol{x},k) = \Phi(k)\Bigg[&\frac{\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}}{(\mathrm{i}kx)^2} j_2\left(\frac{\mathrm{i}x}{k}\right) + \frac{x_i x_j x_m x_n}{(\mathrm{i}kx)^4} j_4\left(\frac{\mathrm{i}x}{k}\right) \\ &+ \frac{x_i x_j\delta_{mn} + x_i x_m\delta_{jn} + x_i x_n\delta_{jm} + x_j x_m\delta_{in} + x_j x_n\delta_{im} + x_m x_n\delta_{ij}}{(\mathrm{i}kx)^3} j_3\left(\frac{\mathrm{i}x}{k}\right)\Bigg], \end{aligned} \tag{A.2e}$$

where $\Phi(k) = -k(k+a)\mathrm{e}^{-a/k}$, $\mathrm{i} = \sqrt{-1}$ and $j_n(\mathrm{i}x/k)$ is the spherical Bessel function of order n. Thus in order to calculate the averages

$$\langle M(k)\rangle_{\mathrm{I}} = \frac{1}{V_\Omega}\int_\Omega M(\boldsymbol{x},k)\mathrm{d}\boldsymbol{x}, \tag{A.3a}$$

$$\langle M_{,i}(k)\rangle_{\mathrm{I}} = \frac{1}{V_\Omega}\int_\Omega M_{,i}(\boldsymbol{x},k)\mathrm{d}\boldsymbol{x}, \tag{A.3b}$$

$$\langle M_{,ij}(k)\rangle_{\mathrm{I}} = \frac{1}{V_\Omega}\int_\Omega M_{,ij}(\boldsymbol{x},k)\mathrm{d}\boldsymbol{x}, \tag{A.3c}$$

$$\langle M_{,ijm}(k)\rangle_{\mathrm{I}} = \frac{1}{V_\Omega}\int_\Omega M_{,ijm}(\boldsymbol{x},k)\mathrm{d}\boldsymbol{x}, \tag{A.3d}$$

$$\langle M_{,ijmn}(k)\rangle_{\mathrm{I}} = \frac{1}{V_\Omega}\int_\Omega M_{,ijmn}(\boldsymbol{x},k)\mathrm{d}\boldsymbol{x}, \tag{A.3e}$$

we only need to evaluate the following six primitive averages, where V_Ω is the volume of the inclusion. Here we give the results directly:

$$\langle j_0(\mathrm{i}x/k)\rangle_{\mathrm{I}} = \frac{4\pi}{V_\Omega}\left[ak^2\cosh\left(\frac{a}{k}\right) - k^3\sinh\left(\frac{a}{k}\right)\right], \tag{A.4a}$$

$$\langle j_1(\mathrm{i}x/k)/x\rangle_{\mathrm{I}} = \frac{4\pi}{V_\Omega}\mathrm{i}k^2\left[\sinh\left(\frac{a}{k}\right) - \mathrm{Shi}\left(\frac{a}{k}\right)\right], \tag{A.4b}$$

$$\langle j_2(\mathrm{i}x/k)/x^2\rangle_{\mathrm{I}} = \frac{2\pi}{V_\Omega}\left[k\,\mathrm{Shi}\left(\frac{a}{k}\right) + \frac{3k^3}{a^2}\sinh\left(\frac{a}{k}\right) - \frac{3k^2}{a}\cosh\left(\frac{a}{k}\right)\right], \tag{A.4c}$$

$$\langle x_m x_n j_2(\mathrm{i}x/k)/x^2\rangle_{\mathrm{I}} = \delta_{mn}\frac{4\pi k^2}{3V_\Omega}\left[4k\sinh\left(\frac{a}{k}\right) - a\cosh\left(\frac{a}{k}\right) - 3k\,\mathrm{Shi}\left(\frac{a}{k}\right)\right], \tag{A.4d}$$

$$\begin{aligned}\langle x_m x_n j_3(\mathrm{i}x/k)/x^3\rangle_{\mathrm{I}} = \delta_{mn}\frac{2\pi}{3V_\Omega}\frac{\mathrm{i}k^2}{a^2}\Big[&15ak\cosh\left(\frac{a}{k}\right)\\ &- (2a^2+15k^2)\sinh\left(\frac{a}{k}\right) - 3a^2\,\mathrm{Shi}\left(\frac{a}{k}\right)\Big],\end{aligned} \tag{A.4e}$$

$$\begin{aligned}\langle x_m x_n x_p x_q j_4(\mathrm{i}x/k)/x^4\rangle_{\mathrm{I}} = {}&(\delta_{mn}\delta_{pq} + \delta_{mp}\delta_{nq} + \delta_{mq}\delta_{np})\\ &\times\frac{2\pi}{15V_\Omega}\frac{k^2}{a^2}\Big[(2a^3+105ak^2)\cosh\left(\frac{a}{k}\right)\\ &- (22a^2k+105k^3)\sinh\left(\frac{a}{k}\right) - 15a^2k\,\mathrm{Shi}\left(\frac{a}{k}\right)\Big],\end{aligned} \tag{A.4f}$$

where a is the radius of the spherical inclusion, $V_\Omega = 4\pi a^3/3$, and $\mathrm{Shi}(z) = \int_0^z \sinh(t)\mathrm{d}t$. With the volume averages of ψ, ϕ and M over the inclusion and their derivatives in hand, the average Eshelby-like tensors can be easily evaluated using the expressions given by Cheng and He (1995).

For convenience, we only list the two components of the average symmetric Eshelby-like tensor appeared in Eq. (28):

$$\langle K^{\mathrm{S}}_{1212}\rangle_{\mathrm{I}} = \frac{3}{5}\frac{(K+2\mu)}{(3K+4\mu)} - \frac{3h(a+h)\kappa}{5a^3(\kappa+2\mu)}\mathrm{e}^{-a/h}\left[a\cosh\frac{a}{h} - h\sinh\frac{a}{h}\right], \tag{A.5a}$$

$$\langle K^{\mathrm{S}}_{iijj}\rangle_{\mathrm{I}} = \frac{9K}{3K+4\mu}, \tag{A.5b}$$

where K, μ, κ, γ are the micropolar moduli appeared in Eq. (7) for the matrix material, and $h = \sqrt{\gamma(2\mu+\kappa)/4\mu\kappa}$. The first term of Eq. (A.5a) is just the counterpart of the classical Eshelby tensor for a spherical inclusion.

References

Aifantis, E.C., 1984. On the microstructural origin of certain inelastic models. Trans. ASME J. Eng. Mater. Technol. 106, 326–330.

Bao, G., Hutchinson, J.W., McMeeking, R.M., 1991. Particle reinforcement of ductile matrices against plastic flow and creep. Acta Metall. Mater. 39, 1871–1882.

Bazant, Z.P., Pijaudier-Cabot, G., 1988. Nonlocal continuum damage, localization instability and convergence. ASME J. Appl. Mech. 55, 287–293.

Chen, S.H., Wang, T., 2001. Strain gradient theory with couple stress for crystalline solids. Eur. J. Mech. A/Solids 20, 739–756.

Cheng, Z.Q., He, L.H., 1995. Micropolar elastic fields due to a spherical inclusion. Int. J. Eng. Sci. 33 (3), 389–397.

Cheng, Z.Q., He, L.H., 1997. Micropolar elastic fields due to a circular cylindrical inclusion. Int. J. Eng. Sci. 35 (7), 659–668.

Christensen, R.M., Lo, K.H., 1979. Solutions for effective shear properties in three phase space and cylinder model. J. Mech. Phys. Solids 27, 315–330.

Cosserat, E.F., 1909. Theorie des Corps Deformables. A. Herman et. Fils, Paris.

Dai, L.H., Ling, Z., Bai, Y.L., 1999. A strain gradient-strengthening law for particle reinforced metal matrix composites. Scripta Mater. 41, 245–251.

De Borst, R., 1993. A generalization of J_2-flow theory for polar continua. Comp. Meth. Appl. Mech. Eng. 103, 347–362.

Eringen, A.C., 1968. Theory of micropolar elasticity. In: Leibowitz, H. (Ed.), Fracture, An Advanced Treatise. Academic Press, New York.

Eshelby, J.D., 1957. The determination of the elastic field of an ellipsoidal inclusion and related problem. Proc. R. Soc. Lond. A 241, 376–396.

Fleck, N.A., Hutchinson, J.W., 1997. Strain gradient plasticity. Adv. Appl. Mech. 33, 295–361.

Fleck, N.A., Muller, G.M., Ashby, M.F., Hutchinson, J.W., 1994. Strain gradient plasticity: theory and experiment. Acta Metall. Mater. 42, 475–487.

Forest, S., Dendievel, R., Canova, G.R., 1999. Estimating the overall properties of heterogeneous Cosserat materials. Modelling Simul. Mater. Sci. Eng. 7, 829–840.

Forest, S., Barbe, F., Cailletaud, G., 2000. Cosserat modeling of size effects in the mechanical behavior of polycrystals and muti-phase materials. Int. J. of Solids Struct. 37, 7105–7126.

Gao, H., Huang, Y., 2001. Taylor-based nonlocal theory of plasticity. Int. J. Solids Struct. 38, 2615–2637.

Gao, H., Huang, Y., Nix, W.D., 1999. Modeling plasticity at the micrometer scale. Naturwissenschaften 86, 507–515.

Hashin, Z., Strikman, S., 1963. A variational approach to the theory of the elastic behavior of multiphase materials. J. Mech. Phys. Solids 11, 127–140.
Hershey, A.V., 1954. The elasticity of an isotropic aggregate of anisotropic cubic crystals. J. Appl. Mech. 21, 226–240.
Hu, G.K., 1996. A method of plasticity for general aligned spheroidal void or fiber-reinforced composites. Int. J. Plasticity 12, 439–449.
Hu, G.K., Weng, G.J., 2000a. Connection between the double inclusion model and the Ponte Castaneda-Willis, Mori–Tanaka, and Kuster–Toksoz model. Mech. Mater. 32, 495–503.
Hu, G.K., Weng, G.J., 2000b. Some reflections on the Mori–Tanaka and Ponte Castaneda–Willis methods with randomly oriented ellipsoidal inclusions. Acta Mech. 140, 31–40.
Hu, G.K., Liu, X.N., Lu, T.J., in press. A variational method for nonlinear micropolar composites. Mech. Mater.
Hwang, K.C., Guo, Y., Jiang, H., Huang, Y., Zhuang, Z., 2004. The finite deformation theory of Taylor-based nonlocal plasticity. Int. J. Plasticity 20, 831–839.
Ji, B., Wang, T.C., 2003. Plastic constitutive behavior of short-fiber/particle reinforced composites. Int. J. Plasticity 19, 565–581.
Kim, H.K., Oh, S.I., 2003. Finite element analysis of grain-by-grain deformation by crystal plasticity with couple stress. Int. J. Plasticity 19, 1245–1270.
Koiter, W.T., 1964. Couple-stresses in the theory of elasticity, I and II. Proc. Roy. Netherlands Acad. Sci. B 67, 17–44.
Kroner, E., 1958. Berechung der elastischen konstanten des vielkristalls aus den konstanten des einkristalls. Z. Phys. 151, 504–518.
Lippmann, H., 1995. Cosserat plasticity and plastic spin. ASME Appl. Mech. Rev., 48:753–762.
Lloyd, D.J., 1994. Particle-reinforced aluminum and magnesium matrix composite. Int. Mater. Rev. 39, 1–23.
Milton, G.W., 1981. Bounds on the electromagnetic, elastic and other properties of two-component composites. Phys. Rev. Lett. 46, 542–545.
Mindlin, R., 1963. Influence of couple-stress on stress concentrations. Exp. Mech. 3, 1–7.
Mindlin, R., 1964. Microstructure in linear elasticity. Arch. Rat. Mech. Anal. 10, 51–78.
Mori, T., Tanaka, K., 1973. Average stress in matrix and average elastic energy of materials with misfitting inclusions. Acta Metall. Mater. 21, 571–574.
Muhlhaus, H.B., Vardoulakis, I., 1987. The thickness of shear bands in granular materials. Geotechnique 37, 271–283.
Mura, T., 1982. Micromechanics of Defects in Solids. Martinus Nijhoff, La Hague.
Nan, C.W., Clarke, D.R., 1996. The influence of particle size and particle fracture on the elastic/plastic deformation of metal matrix composites. Acta Mater. 44, 3801–3811.
Nemat-Nasser, S., Hori, M., 1993. Micromechanics: Overall Properties of Heterogeneous Materials. Elsevier, North-Holland.
Nowacki, W., 1986. Theory of Asymmetric Elasticity. Pergamon press, New York.
Ostoja-Starzewski, M., Boccara, S.D., Jasiuk, I., 1999. Couple stress moduli and characteristic length of a two phase composite. Mech. Res. Commun. 26, 387–396.
Ponte Castañeda, P., 1991. The effective mechanical properties of nonlinear isotropic composite. J. Mech. Phys. Solids 39, 45–71.
Qiu, Y.P., Weng, G.J., 1992. A theory of plasticity for porous materials and particle-reinforced composites. Int. J. Plasticity 59, 261–268.
Rashid, K., George, Z., 2004. Analytical and experimental determination of the material intrinsic length scale of strain gradient plasticity theory from micro- and nano-indentation experiments. Int. J. Plasticity 20, 1139–1182.
Roy, A.B., Marc, G.D., Frank, P.T., 2003. Nonlocal implicit gradient-enhanced elasto-plasticity for the modelling of softening behaviour. Int. J. Plasticity 19, 403–433.
Shu, J.Y., Barlow, C.Y., 2000. Strain gradient effects on microscopic strain fields in a metal matrix composite. Int. J. Plasticity 16, 563–591.
Stolken, J.S., Evans, A.G., 1998. A microbend test method for measuring the plasticity length scale. Acta Mater. 46, 5109–5115.

Suquet, P., 1995. Overall properties of nonlinear composites: a modified secant moduli theory and its link with Ponte Castaneda's nonlinear variational procedure. C.R. Acad. Sci. Paris Ser. II 320, 563–571.

Tandon, G.P., Weng, G.J., 1988. A theory of particle-reinforced plasticity. ASME J. Appl. Mech. 55, 126–135.

Torquato, S., 1991. Random heterogeneous media: microstructure and improve bounds on effective properties. Appl. Mech. Rev. 44, 37–75.

Toupin, R., 1962. Elastic material with couple-stresses. Arch. Rational Mech. Anal. 11, 385–414.

Vardoulakis, I., 1989. Shear-banding and liquefaction in granular materials on the basis of a Cosserat continuum theory. Ing. Arch. 59, 106–113.

Wang, W., Huang, Y., Hsia, K.J., Hu, K.X., Chandra, A., 2003. A study of microbend test by strain gradient plasticity. Int. J. Plasticity 19, 365–382.

Wei, Y.G., 2001. Particulate size effects in the particle-reinforced metal–matrix composites. Acta Mech. Sinica 17:1, 45–58.

Xun, F., Hu, G.K., Huang, Z.P., 2004. Effective in-plane moduli of composite with a micropolar matrix and coated fibers. Int. J. Solids Struct. 41, 247–265.

Yang, J., 2000. Micromechanical modeling of mechanical properties for TiAl Alloys. Master Thesis, Beijing Institute of Technology (In chinese).

Yang, J., Cady, C., Hu, M.S., et al., 1990. Effects of damage on the flow strength and ductility of a ductile Al-alloy reinforced with SiC particulates. Acta Metall. Mater. 38, 2613–2619.