Pergamon

Int. J. Solids Structures Vol. 34, No. 8, pp. 1007–1015, 1997

0020–7683/97 $17.00 + .00

PII: S0020–7683(96)00044–3

COMPOSITE PLASTICITY BASED ON MATRIX AVERAGE SECOND ORDER STRESS MOMENT

GENGKAI HU
Department of Applied Mechanics, Beijing Institute of Technology 100081, Beijing, China

(*Received* 18 *January* 1995; *in revised form* 14 *March* 1996)

Abstract—An effective stress of a ductile matrix is defined directly from the average second order stress moment. It is evaluated exactly provided that an estimation of the composite effective moduli is given. On the basis of this effective stress and the secant moduli concept originally proposed by Berveiller and Zaoui (Berveiller, M. and Zaoui, A. (1979). An extension of the self-consistent scheme to plastically-flowing polycrystals. *J. Mech. Phys. Solids* **35**, 325–344) and modified for composite materials by Tandon and Weng (Tandon, G. P. and Weng, G. J. (1988). A theory of particle reinforced plasticity. *ASME J. Appl. Mech.* **55**, 126–135), a method for composite plasticity is then proposed. The method is capable of predicting the influence of hydrostatic stress on particle-reinforced composite yielding, especially for porous materials at high triaxiality. Compared to Tandon and Weng's model, the proposed method gives always softer predictions. For particle reinforced composites, the new matrix average effective stress coincides with that obtained by Ponte Castaneda's variational procedure (Ponte Castaneda, P. (1991). The effective mechanical properties of nonlinear isotropic composites. *J. Mech. Phys. Solids* **39**, 45–71).

1. INTRODUCTION

There are, in general, three classes (not exhaustive) of methods for predicting nonlinear composite properties. The first is the numerical method based on the periodic microstructure assumptions (Tvergaard, 1990, Gao *et al.*, 1991). The second is the variational methods which can develop the bounds for the effective behavior of nonlinear composites (Talbot and Willis, 1992, Willis, 1991, Ponte Castaneda, 1991 and Suquet, 1993). The work of Ponte Castaneda (1991) makes use of arbitrary bounds and estimates for classes of linear comparison composites to generate bounds and estimates for the corresponding classes of nonlinear composites. Application of this method was performed by Li and Ponte Castaneda (1993) for composites made of two elastically incompressible phases. The third method is originally proposed by Berveiller and Zaoui (1979) and modified for composite materials by Tandon and Weng (1988). This model makes use of a linear comparison material, whose matrix elastic moduli at every instant are chosen to coincide with the average secant moduli of the matrix to characterize its elastoplastic state. The method generates directly elastic results to plastic cases with the help of secant moduli of the composite. This model is simple and straightforward for applications. However, as recognized by Qiu and Weng (1992), under a pure hydrostatic load, Tandon and Weng's original model always predicts elastic behavior for the composites reinforced by spherical inclusions. In their model the effective stress of the matrix is evaluated from the average stress of the matrix, and the local stress variation in the matrix is not taken into account. Recently, Qiu and Weng (1992) redefined an effective stress of the matrix from an energy approach to improve the predicted results. However, in their model the effective stress can not be obtained exactly.

On the basis of the secant moduli concept (Berveiller and Zaoui, 1979, Tandon and Weng, 1988), we propose a method for composite plasticity. The effective stress of the matrix will be defined from the average second order stress moment. It will be calculated in an exact manner from the linear composite effective moduli. Particle-reinforced composites will be analyzed in detail. Finally, the possible connections with Ponte Castaneda's work (1991) will be also discussed.

2. AVERAGE SECOND ORDER STRESS MOMENT

Here we will follow the general method proposed by Bobeth and Diener (1986, 1987) and Kreher (1990). Considering a representative volume V, a uniform macroscopic stress Σ_{ij} is prescribed along its boundary. Corresponding local stress and strain are denoted by $\sigma_{ij}(x)$, $\varepsilon_{ij}(x)$. They satisfy:

$$\begin{aligned} \sigma_{ij,j} &= 0 \\ \varepsilon_{ij} &= \tfrac{1}{2}(u_{i,j}+u_{j,i}) \end{aligned} \tag{1}$$

where u_i is the displacement field. Along the boundary of the representative volume, we have $\sigma_{ij}n_j = \Sigma_{ij}n_j$. It is easy to show that:

$$\langle \sigma_{ij} \rangle = \Sigma_{ij}. \tag{2}$$

$\langle A \rangle$ is the volume of quantity A over the whole representative volume.

In the case of elasticity, the average of the stored elastic energy of the composite is:

$$\tfrac{1}{2}\langle \sigma_{ij}\varepsilon_{ij} \rangle = U. \tag{3}$$

Using the Hill condition (Hill, 1963), we get

$$2U = \langle \sigma_{ij}\varepsilon_{ij} \rangle = \langle \sigma_{ij}s_{ijkl}\sigma_{kl} \rangle = \Sigma_{ij}S_{ijkl}\Sigma_{kl} \tag{4}$$

where s_{ijkl}, S_{ijkl} are local and composite effective compliance tensors, respectively.

Now, under a constant applied macroscopic load, a variation of the local compliance tensor δs_{ijkl} will lead to a variation of the local stress $\delta\sigma_{ij}$ in turn a variation of the average stored energy δU and the composite effective compliance tensor δS_{ijkl}. We have:

$$\Sigma_{ij}\,\delta S_{ijkl}\Sigma_{kl} = \langle \sigma_{ij}\,\delta s_{ijkl}\sigma_{kl} \rangle + 2\langle \sigma_{ij}s_{ijkl}\,\delta\sigma_{kl} \rangle. \tag{5}$$

Since, under a constant applied stress, the volume average of the local stress variation vanishes, we get:

$$\Sigma_{ij}\,\delta S_{ijkl}\Sigma_{kl} = \langle \sigma_{ij}\,\delta s_{ijkl}\sigma_{ij} \rangle. \tag{6}$$

Again the Hill condition is used.

In general, the exact composite effective compliance S_{ijkl} is not available. In the following, the approximations or bounds for the effective moduli of linear composite materials will be used to calculate the matrix effective stress.

3. COMPOSITE WITH SPHERICAL ISOTROPIC INCLUSIONS

3.1. *Preliminary*

The secant moduli method (Berveiller and Zaoui, 1979, Tandon and Weng, 1988) will be used to account for the decreasing constraint power of matrix plastic deformation. The overall stress–strain relation of the nonlinear composite is characterized by the effective moduli of a series of linear elastic comparison composites. The matrix moduli of the linear comparison composites are taken as the secant moduli of the nonlinear matrix at the current deformation state. The effective stress and strain of the nonlinear matrix is assumed to follow the modified Ludwik's equation

$$\sigma_y = \sigma_{y0} + h(\varepsilon_e^p)^n \tag{7}$$

where σ_{y0}, h and n are initial yield stress, strength coefficient, and work-hardening exponent. ε_e^p is usual effective plastic strain.

The secant shear and bulk moduli of the matrix at the plastic strain ε_{ij}^p is defined by

$$\mu_0^s = 1/(1/\mu_0 + 3\varepsilon_e^p/(\sigma_y + h(\varepsilon_e^p)^n)), \quad k_0^s = k_0 \tag{8}$$

where μ_0, k_0 are the usual elastic shear and bulk moduli.

With the help of the matrix secant shear and bulk moduli, the effective shear and the bulk moduli of the linear comparison composite can be estimated by the Mori-Tanaka mean field theory (Tandon and Weng, 1988):

$$\frac{k_c}{k_0} = 1 + \frac{c_1(k_1 - k_0)}{c_0\alpha(k_1 - k_0) + k_0} \tag{9}$$

$$\frac{\mu_c}{\mu_0} = 1 + \frac{c_1(\mu_1 - \mu_0)}{c_0\beta(\mu_1 - \mu_0) + \mu_0} \tag{10}$$

where $\alpha = 1/(1 + 4u_0/3k_0)$ and $\beta = 2/15(3 + 18u_0/3k_0)/(1 + 4u_0/3k_0)$. So as not to complicate the notation, we keep using μ_0, k_0 for the matrix secant shear and bulk moduli.

To construct nonlinear composite stress and strain relations, we have to know, under a given applied macroscopic load, the corresponding linear comparison composite, or, more concretely, the matrix secant shear modulus of the studied nonlinear composite. This will be established in the following section by defining a new matrix effective stress.

3.2. *Matrix average effective stress of the comparison composite*

For a particle reinforced composite, the composite as a whole is isotropic. We split the stress tensor into a deviatoric and a spherical part:

$$\begin{aligned} \sigma_{ij} &= \sigma'_{ij} + \sigma\delta_{ij} \\ \Sigma_{ij} &= \Sigma'_{ij} + \Sigma\delta_{ij}. \end{aligned} \tag{11}$$

The shear and bulk moduli of the inclusions, the matrix and the linear comparison composite are denoted, respectively, by u_1, k_1, u_0, k_0 and u_c, k_c. Let only the shear modulus of the matrix undergo a small variation, and the other local material constants be kept constant. In this case, eqn (6) reduces to

$$\frac{V_0}{V}\langle\sigma'_{ij}\sigma'_{ij}\rangle_0\,\delta\!\left(\frac{1}{2u_0}\right) = \Sigma'_{ij}\Sigma'_{ij}\,\delta\!\left(\frac{1}{2u_c}\right) + \Sigma^2\,\delta\!\left(\frac{1}{k_c}\right)$$

we get

$$\langle\sigma'_{ij}\sigma'_{ij}\rangle_0 = \frac{1}{c_0}\left\{\left(\frac{u_0}{u_c}\right)^2\frac{\partial u_c}{\partial u_0}\Sigma'_{ij}\Sigma'_{ij} + 2\left(\frac{u_0}{k_c}\right)^2\frac{\partial k_c}{\partial u_0}\Sigma^2\right\} \tag{12}$$

where $\langle A\rangle_0$ is the volume average of quantity A over the matrix, c_0 is the volume fraction of the matrix. Equation (8) will give the same yield condition as Li and Ponte Castaneda (1994) through a variational procedure.

In the same manner, we obtain for the spherical stress

$$\langle\sigma^2\rangle_0 = \frac{1}{c_0}\left\{\frac{1}{2}\left(\frac{k_0}{u_c}\right)^2 \frac{\partial u_c}{\partial k_0}\Sigma'_{ij}\Sigma'_{ij} + \left(\frac{k_0}{k_c}\right)^2 \frac{\partial k_c}{\partial k_0}\Sigma^2\right\}. \tag{13}$$

We observe that if the comparison composite effective shear and bulk moduli are given, the average second order stress moment of the matrix can then be obtained.

With the help of eqns (9) and (10) for the linear comparison composite, the matrix average effective stress is obtained:

$$\langle\sigma_e^2\rangle_0 = \frac{\Sigma_e^2}{A^2} + \frac{\Sigma^2}{B^2} \tag{14}$$

and

$$A^2 = \frac{[c_0(\beta-1)(u_1-u_0)+u_1]^2}{(u_1-u_0)^2[c_0\beta^2 - c_0\beta - 1/5c_1\alpha(1-\alpha)] + (u_1^2-u_0^2)\beta + u_0^2}$$

$$B^2 = \frac{4}{9}\frac{[c_0(\alpha-1)(k_1-k_0)+k_1]^2}{(1-\alpha)^2 c_1(k_1-k_0)^2}$$

where $\sigma_e^2 = 3/2\sigma'_{ij}\sigma'_{ij}$ and $\Sigma_e^2 = 3/2\Sigma'_{ij}\Sigma'_{ij}$.

If the matrix satisfies the von Mises yield criterion, the yield function of the composite material can be obtained by setting $\langle\sigma_e^2\rangle_0 = \sigma_y^2$, where σ_y is the current yield stress of the matrix. Since $1/B^2$ is, in general, not zero, so the matrix effective stress based on the average second order stress moment is not zero even under a pure hydrostatic load.

Equation (14) and the matrix hardening law (eqn (7)) together provide a relation between the matrix secant shear modulus and the applied load, so the nonlinear composite stress and strain can be constructed. In the following, porous materials and rigid inclusions reinforced composites will be examined in detail.

3.3. *Porous materials or rigid inclusions reinforced composites*

Voided materials. In this case setting $\mu_1 = k_1 = 0$, we obtain from eqn (14) the average effective stress of the matrix:

$$\langle\sigma_e^2\rangle = \frac{1}{c_0^2(\beta-1)^2}\left[(1-\beta)(1-c_0\beta) - \frac{1}{5}\alpha(1-\alpha)\right]\Sigma_e^2 + \frac{9c_1}{4c_0^2}\Sigma^2. \tag{15}$$

The yield function of the composite can be simply obtained by setting $\langle\sigma_e^2\rangle_0 = \sigma_y^2$, where σ_y is the effective stress representing the current flow stress of the matrix.

$$F = \left(\frac{\Sigma_e}{\sigma_y}\right)^2 + \frac{9c_1(\beta-1)^2}{4[(1-\beta)(1-c_0\beta)-\frac{1}{5}\alpha(1-\alpha)]}\left(\frac{\Sigma}{\sigma_y}\right)^2 - \frac{c_0^2(\beta-1)^2}{[(1-\beta)(1-c_0\beta)-\frac{1}{5}\alpha(1-\alpha)]} = 0. \tag{16}$$

This yield function of the composite for an elastically compressible matrix is different from that given by Qiu and Weng (1993). The difference is probably due to the neglected terms by Qiu and Weng, since in our case the exact average effective stress of the matrix is obtained.

For an elastically incompressible matrix $\alpha = 1$, $\beta = 2/5$, the yield function and the average effective stress of the matrix become:

$$F=\left(\frac{\Sigma_e}{\sigma_y}\right)^2+\frac{9c_1}{4(1+\frac{2}{3}c_1)}\left(\frac{\Sigma}{\sigma_y}\right)^2-\frac{c_0}{1+\frac{2}{3}c_1}=0 \tag{17}$$

$$\langle\sigma_e^2\rangle_0=\frac{1}{c_0^2}\left[\left(1+\frac{2}{3}c_1\right)\Sigma_e^2+\frac{9c_1}{4}\Sigma^2\right]. \tag{18}$$

This result coincides with those obtained by Qiu and Weng (1992), in turn this method corresponds to the Ponte Castaneda's lower bound approach (Ponte Castaneda, 1991) as demonstrated by Qiu and Weng (1992).

Rigid inclusions. In this case $\mu_1, k_1 -> \infty$, the average effective stress of the matrix and the yield function of the composite become:

$$\langle\sigma_e^2\rangle_0=\frac{1}{(c_0\beta+c_1)^2}\left[\beta(c_0\beta+c_1)-\frac{1}{5}c_1\alpha(1-\alpha)\right]\Sigma_e^2+\frac{9c_1}{4}\left(\frac{1-\alpha}{c_0\alpha+c_1}\right)^2\Sigma^2 \tag{19}$$

$$F=\left(\frac{\Sigma_e}{\sigma_y}\right)^2+\frac{9c_1}{4[\beta(c_0\beta+c_1)-\frac{1}{5}c_1\alpha(1-\alpha)]}\left(\frac{(1-\alpha)(c_0\beta+c_1)}{c_0\alpha+c_1}\right)^2\left(\frac{\Sigma}{\sigma_y}\right)^2-\frac{(c_0\beta+c_1)^2}{[\beta(c_0\beta+c_1)-\frac{1}{5}c_1\alpha(1-\alpha)]}=0. \tag{20}$$

For an elastically incompressible matrix $\alpha = 1$, $\beta = 2/5$, the previous results reduce to

$$\langle\sigma_e^2\rangle_0=\frac{2}{2+3c_1}\Sigma_e^2 \tag{21}$$

$$F=\left(\frac{\Sigma_e}{\sigma_y}\right)^2-\frac{2+3c_1}{2}=0. \tag{22}$$

Now we will examine the difference between the proposed matrix effective stress and that given by Tandon and Weng (1988), that is $\bar{\sigma}_e^2 = 3/2\langle\sigma'_{ij}\rangle_0\langle\sigma'_{ij}\rangle_0$. The compared results are depicted in Fig. 1. We note that the higher the triaxiality $\omega = \Sigma/\Sigma_e$ and the mismatch of the shear modulus of the particles and matrix, the greater the difference.

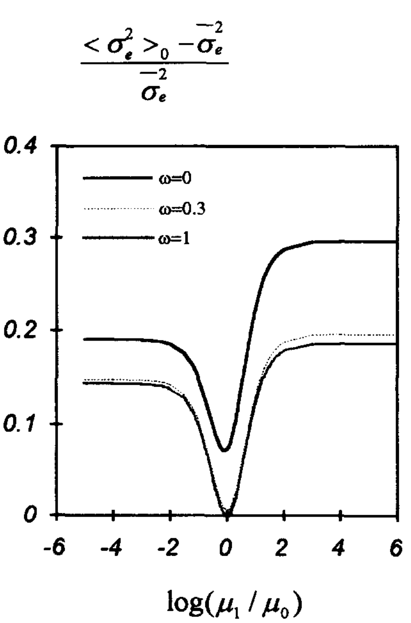

Fig. 1. Relative difference of the matrix effective stress predicted by Tandon and Weng's original model and by the present model ($k_1/k_0 = 10$, $\mu_0/k_0 = 0.383$, $c_1 = 20\%$).

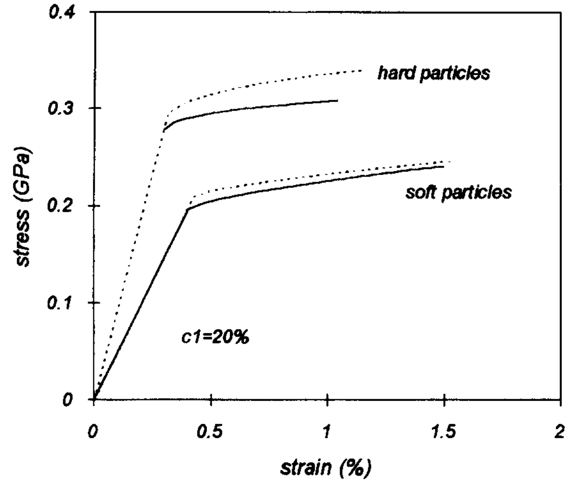

Fig. 2. Comparison between the present method (solid line) and Tandon and Weng's model (dashed line) for particle-reinforced composites.

Figure 2 gives the comparison results between the above method and Tandon and Weng's model. The material properties for a Carbide/Aluminum system (hard particles) are:

$$E_0 = 68.3\text{ GPa},\quad \nu_0 = 0.33,\quad \sigma_{y0} = 250\text{ MPa},$$
$$h = 173\text{ MPa},\quad n = 0.455,\quad E_1 = 490\text{ GPa},\quad \nu_1 = 0.17.$$

For the soft particles composite system, the particle properties are $E_1 = 6.83$ GPa, $\nu_1 = 0.33$ and the matrix's properties remain the same.

The dash lines are the results predicted by Tandon and Weng's model and the solid lines are the results obtained by the proposed method. It is noted that the proposed method gives always softer predictions compared to the Tandon and Weng's model.

4. DISCUSSIONS

In this section, we will further explore possible connections between the proposed method and that given by Ponte Castaneda (1991). Now we consider a composite made of two incompressible isotropic phases, inclusions are of spheroidal shape and well aligned (the symmetry axis is taken to be $x1$). The composite as a whole is also incompressible. This problem has been analyzed by Li and Ponte Castaneda (1993) using the variational method. In this section we will reconsider this problem by the above proposed method. The composite material is characterized by three shear moduli μ_d^c, μ_p^c, μ_n^c and the composite energy density is (Li and Ponte Castaneda, 1993):

$$U = \frac{1}{2\mu_d^c}\bar{\tau}_d^2 + \frac{1}{2\mu_p^c}\bar{\tau}_p^2 + \frac{1}{2\mu_n^c}\bar{\tau}_n^2 \tag{23}$$

where $\bar{\tau}_d$, $\bar{\tau}_p$ and $\bar{\tau}_n$ are the three transversely isotropic invariant of the applied stress tensor Σ_{ij}. Their relations are

$$\begin{aligned}\bar{\tau}_d &= \frac{1}{\sqrt{3}}\left[\frac{1}{2}(\Sigma_{22}+\Sigma_{33})-\Sigma_{11}\right]\\ \bar{\tau}_p &= [\Sigma_{23}^2+\tfrac{1}{4}(\Sigma_{33}-\Sigma_{22})^2]^{1/2}\\ \bar{\tau}_n &= (\Sigma_{12}^2+\Sigma_{13}^2)^{1/2}.\end{aligned} \tag{24}$$

The same relations apply for the corresponding shear strain. When the matrix undergoes a plastic deformation, we will use its secant moduli to characterize its plastic state. The previous composite shear moduli should be taken as the corresponding secant shear moduli of the composite. The stress–strain relations of the composite are related by its shear moduli as the following:

$$\bar{\tau}_d = 2\mu_d^c\bar{\varepsilon}_d, \quad \bar{\tau}_p = 2\mu_p^c\bar{\varepsilon}_p, \quad \bar{\tau}_n = 2\mu_n^c\bar{\varepsilon}_n. \tag{25}$$

The composite shear moduli (secant) depend on the applied load through the secant shear moduli of the matrix. To determine their relation, we must calculate the average effective stress of the matrix. Using the concept introduced in Section 2, we obtain

$$\langle\sigma'_{ij}\sigma'_{ij}\rangle_0 = \frac{2\mu_0^2}{c_0}\left[\frac{1}{\mu_d^{c2}}\frac{\partial\mu_d^c}{\partial\mu_0}\bar{\tau}_d^2 + \frac{1}{\mu_p^{c2}}\frac{\partial\mu_p^c}{\partial\mu_0}\bar{\tau}_p^2 + \frac{1}{\mu_n^{c2}}\frac{\partial\mu_n^c}{\partial\mu_0}\bar{\tau}_n^2\right]. \tag{26}$$

For the composites made of two elastically incompressible isotropic phases, its shear moduli are estimated as the following (Li and Ponte Castaneda, 1993):

$$\begin{aligned}
\frac{\mu_0}{\mu_d^c} &= 1 - \frac{c_1(1-\varepsilon)}{1-c_0(1-\varepsilon)f(r)} \\
\frac{\mu_0}{\mu_p^c} &= 1 - \frac{c_1(1-\varepsilon)}{1-c_0(1-\varepsilon)(1-2\Pi_{2323})} \\
\frac{\mu_0}{\mu_n^c} &= 1 - \frac{c_1(1-\varepsilon)}{1-c_0(1-\varepsilon)(1-2\Pi_{1212})}
\end{aligned} \tag{27}$$

$f(r)$, Π_{2323} and Π_{1212} were given by Li and Ponte Castaneda (1993) (independent of μ_0), $\varepsilon = \mu_0/\mu_1$.

The effective shear stress of the matrix can be obtained by using eqns (26) and (27):

$$\tau_e^2 = \tfrac{1}{2}\langle s_{ij}s_{ij}\rangle_0 = H_d\bar{\tau}_d^2 + H_p\bar{\tau}_p^2 + H_n\bar{\tau}_n^2 \tag{28}$$

where

$$\begin{aligned}
H_d &= \frac{1}{c_0}\left\{1 - c_1\frac{1-c_0(1-\varepsilon)^2 f(r)}{[1-c_0(1-\varepsilon)f(r)]^2}\right\} \\
H_p &= \frac{1}{c_0}\left\{1 - c_1\frac{1-c_0(1-\varepsilon)^2(1-2\Pi_{2323})}{[1-c_0(1-\varepsilon)(1-2\Pi_{2323})]^2}\right\} \\
H_n &= \frac{1}{c_0}\left\{1 - c_1\frac{1-c_0(1-\varepsilon)^2(1-2\Pi_{1212})}{[1-c_0(1-\varepsilon)(1-2\Pi_{1212})]^2}\right\}.
\end{aligned}$$

We note that, for linear comparison composites, the determined matrix average effective shear stress coincides with that obtained by Li *et al.* (1993). For such a composite in the case of a uniaxial loading, Fig. 3 shows the comparison results between the proposed method and that based on Tandon and Weng's model (the matrix and reinforced phase are incompressible, other material constants are the same as the previous). As in the case of the particle-reinforced composite, the secant moduli method based on the matrix average second order stress moment gives always softer prediction compared to Tandon and Weng's model.

The previous results may suggest a more general correspondence between the secant moduli method based on the second order stress moment and the variational method by Ponte Castaneda (1991). In the author's later work, this connection was established for a

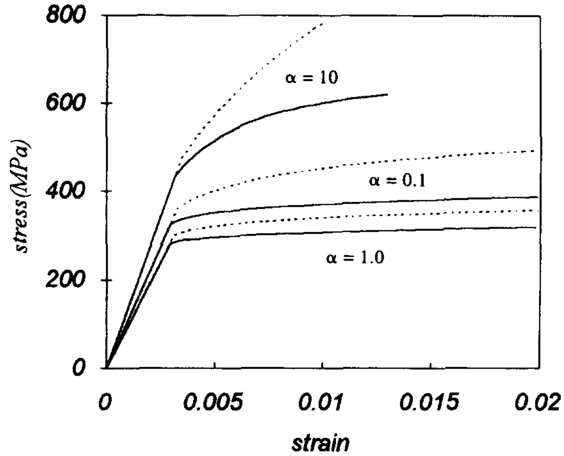

Fig. 3. Comparison between the present method (solid line) and Tandon and Weng's model (dashed line) for two phase incompressible aligned composite ($c_1 = 20\%$).

two phase composite, and in the revision of this paper, the author is noted that Suquet (1995) established independently this connection through the strain field for a more general composite. So the secant moduli method based on the second order stress moment gives always the low bound estimation proposed by Ponte Castaneda (1991) for nonlinear composites.

5. CONCLUSIONS

On the basis of the secant moduli concept, a method for composite plasticity is proposed. The method makes use of a matrix effective stress derived directly from the average second order stress moment. The proposed method is capable of predicting the influence of hydrostatic stress on the particle-reinforced composite yielding, especially for porous materials at high triaxiality. For the particle reinforced composite, the new matrix average effective stress coincides with those obtained by the variational method (Ponte Castaneda, 1991).

REFERENCES

Bao, G., Hutchinson, J. W. and McMeeking, R. M. (1991) Particle reinforcement of ductile matrices against plastic flow and creep. *Acta Metall. Mat.* **39,** 1871–1882.

Berveiller, M. and Zaoui, A. (1979) An extension of the self-consistent scheme to plastically-flowing polycrystals. *J. Mech. Phys. Solids* **35,** 325–344.

Bobeth, M. and Diener, G. (1986) Field fluctuations in multicomponent mixtures. *J. Mech. Phys. Solids* **34,** 1–17.

Bobeth, M. and Diener, G. (1987) Static elastic and thermoelastic field fluctuation in multiphase composites. *J. Mech. Phys. Solids* **35,** 137–149.

Kreher, W. (1990) Residual stresses and stored elastic energy of composites and polycrystals. *J. Mech. Phys. Solids* **38,** 115–128.

Li, G. and Ponte Castaneda, P. (1993) The effect of particle shape and stiffness on the constitutive behavior of metal-matrix composites. *Int. J. Solids Struct.* **30,** 3189–3209.

Li, G. and Ponte Castaneda, P. (1994) *Appl. Mech. Rev.* **47,** 77–94.

Ponte Castaneda, P. (1991) The effective mechanical properties of nonlinear isotropic composites. *J. Mech. Phys. Solids* **39,** 45–71.

Qiu, Y. P. and Weng, G. J. (1991) The influence of inclusion shape on the overall elastoplastic behavior of a two-phase isotropic composite. *Int. J. Solids Struct.* **27,** 1537–1550.

Qiu, Y. P. and Weng, G. J. (1992) A theory of plasticity for porous materials and particle-reinforced composites. *ASME J. Appl. Mech.* **59,** 261–268.

Qiu, Y. P. and Weng, G. J. (1993) Plastic potential and yield function of porous materials with aligned and randomly oriented spheroidal voids. *Int. J. Plasticity* **9,** 271–290.

Suquet, P. M. (1993) Overall potentials and extremal surfaces of power law or ideally plastic composites. *J. Mech. Phys. Solids* **41,** 981–1002.

Suquet, P. M. (1995) Overall properties of nonlinear composites: a modified secant moduli theory and its link with Ponte Castaneda's nonlinear variational procedure. *C. R. Acad. Sci. Paris III* **320,** 563–571.

Talbot, D. R. S. and Willis, J. R. (1992) Some explicit bounds for the overall behavior of nonlinear composites. *Int. J. Solids Struct.* **29,** 1981–1987.

Tandon, G. P. and Weng, G. J. (1988) A theory of particle-reinforced plasticity. *ASME J. Appl. Mech.* **55,** 126–135.

Tvergaard, V. (1990) Analysis of tensile properties for a whisker-reinforced metal-matrix composite. *Acta Metall. Mat.* **38,** 185–194.

Willis, J. R. (1991) On methods for bounding the overall properties of nonlinear composites. *J. Mech. Phys. Solids* **39,** 73–86.