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The connections between the double-inclusion model and the Ponte Castaneda–Willis, Mori–Tanaka, and Kuster–Toksoz models

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Abstract

In this paper, it is shown that the double-inclusion model (Hori, M., Nemat-Nasser, S., 1993. Double-inclusion model and overall moduli of multi-phase composites. Mech. Mater. 14, 189–206) carries more theoretical connections with other micromechanical models than what is presently realized. In the past, only connections with the Mori–Tanaka (MT) model (Mori, T., Tanaka, K., 1973. Average stress in matrix and average elastic energy of materials with misfitting inclusions. Acta Metall. 21, 571–574) and the self-consistent model (Hill, R., 1965. A self-consistent mechanics of composite materials. J. Mech. Phys. Solids 13, 213–222; Budiansky, B., 1965. On the elastic moduli of some heterogeneous material. J. Mech. Phys. Solids, 13, 223–227) for *aligned* inclusions have been established. By choosing the shape and the relative orientation of the inclusion and the matrix judiciously, the double-inclusion model can produce results for a two-phase composite containing *randomly* oriented ellipsoidal inclusions for the Ponte Castaneda–Willis (PCW) model (Ponte Castaneda, P., Willis, J.R., 1995. The effect of spatial distribution on the effective behavior of composite materials and cracked media. J. Mech. Phys. Solids 43, 1919–1951), MT model, and Kuster–Toksoz (KT) model (Kuster, G.T., Toksoz, M.N., 1974. Velocity and attenuation of seismic waves in two-phase media: I Theoretical formulation. Geophysics, 39, 587–606). These connections have also shed some light into the possible microgeometries for the MT and KT models. The microstructure for the PCW model is already known, and it is now established that the outer shape and orientation of the double inclusion is exactly the spatial distribution ellipsoid of the PCW model. The result also proves that the KT model, widely used in the geophysics community, actually provides a result that is identical to the PCW model and, thus, has a well-defined microstructure that was previously said to be non-existent. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Double inclusion; Elastic moduli; Two-phase composites; Micromechanical models

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1. Introduction

The double-inclusion model as proposed by Nemat-Nasser and Hori (1993) and Hori and Nemat-Nasser (1993) consists of an ellipsoidal inclusion embedded in another ellipsoidal matrix, that is, further embedded in an infinitely extended homogeneous medium. The shape and orientation of the inclusion and the matrix, and the elastic properties of these three phases are arbitrary; as such, it provides great flexibility and can be used to evaluate the effective properties or to construct various micromechanics models of a two-phase composite with different inclusion arrangements. By choosing the inclusion and the matrix to be aligned and identically shaped, and by taking the elastic moduli of the reference medium to be identical to those of the matrix and the effective composite, respectively, they have demonstrated that it can produce the effective moduli of a two-phase composite with *aligned* inclusions for the Mori and Tanaka (1973) model and the self-consistent model (Hill, 1965; Budiansky, 1965). These connections have given some alternative interpretations for these two approaches. Its flexibility also points to the possibility that connections with other micromechanical models may still exist. Establishment of such inter-relations is quite desirable for it could uncover the similarities of the underlying principles, and it can also provide some insights into the possible microgeometries for those approximate models which have been derived from the embedding process of Eshelby (1957). To this end, we decided to look into the double-inclusion model one more time, and consider its possible connections with three useful explicit schemes for the 3-D randomly oriented case: the Ponte Castaneda and Willis (1995) model, the Mori and Tanaka (1973) model, and the Kuster and Toksoz (1974) model. Of course connection for the 3-D random orientation of inclusions automatically implies the connection for the 1-D aligned case.

2. The double-inclusion model

To facilitate our analysis, the double-inclusion model will be recapitulated first. This method was proposed to better account for the interaction between the inclusion and the matrix, and between the inclusion and the inclusion. The inclusions have an ellipsoidal form with a common aspect ratio, and they can be aligned or randomly oriented in a plane or in the space. The modulus tensor of the matrix is denoted by $\boldsymbol{L}_0$, and by $\boldsymbol{L}_1$ for the inclusions, and the volume concentration of the inclusions (phase 1) by c_1.

In order to evaluate the average stress and strain concentration tensors for the matrix and the inclusions, a double-cell V is taken for each inclusion, with both the double-cell and the inclusions being ellipsoidal in shape. The double-cell contains the matrix phase (Γ) and an inclusion (Ω) ($V = \Gamma + \Omega$), and the volume concentration of the inclusion in the double-cell is set equal to c_1. Now, the double-cell is placed into a homogeneous medium with a modulus tensor $\boldsymbol{L}$ under a remote uniform strain $\boldsymbol{\varepsilon}^\infty$, as shown in Fig. 1.

With the introduction of an eigenstrain distribution and the local consistency condition, and then taken over the respective volume average, the average eigenstrain in the inclusion and the matrix region can be estimated. Hori and Nemat-Nasser (1993) found that the average eigenstrain tensors of the inclusion and the matrix satisfy the following relations:

$$\begin{aligned}
&\boldsymbol{L}_1\left[\boldsymbol{\varepsilon}^\infty + \boldsymbol{S}^\Omega\boldsymbol{\varepsilon}_1^* + (\boldsymbol{S}^V - \boldsymbol{S}^\Omega)\boldsymbol{\varepsilon}_2^*\right] = \boldsymbol{L}\left[\boldsymbol{\varepsilon}^\infty + (\boldsymbol{S}^\Omega - \boldsymbol{I})\boldsymbol{\varepsilon}_1^* + (\boldsymbol{S}^V - \boldsymbol{S}^\Omega)\boldsymbol{\varepsilon}_2^*\right],\\
&\boldsymbol{L}_0\left[\boldsymbol{\varepsilon}^\infty + \boldsymbol{S}^V\boldsymbol{\varepsilon}_2^* + \frac{c_1}{1-c_1}(\boldsymbol{S}^V - \boldsymbol{S}^\Omega)(\boldsymbol{\varepsilon}_1^* - \boldsymbol{\varepsilon}_2^*)\right] = \boldsymbol{L}\left[\boldsymbol{\varepsilon}^\infty + (\boldsymbol{S}^V - \boldsymbol{I})\boldsymbol{\varepsilon}_2^* + \frac{c_1}{1-c_1}(\boldsymbol{S}^V - \boldsymbol{S}^\Omega)(\boldsymbol{\varepsilon}_1^* - \boldsymbol{\varepsilon}_2^*)\right],
\end{aligned} \tag{1}$$

where $\boldsymbol{\varepsilon}_1^*$, $\boldsymbol{\varepsilon}_2^*$ are the average eigenstrains introduced in the inclusion and matrix regions so that $\boldsymbol{L}_1$ and $\boldsymbol{L}_0$ can be replaced by $\boldsymbol{L}$ to yield the same average stress and strain in both regions. Furthermore $\boldsymbol{S}^V$, $\boldsymbol{S}^\Omega$ are the Eshelby tensors for the double-cell and the inclusion, respectively, and $\boldsymbol{I}$ is the fourth-order unit tensor.

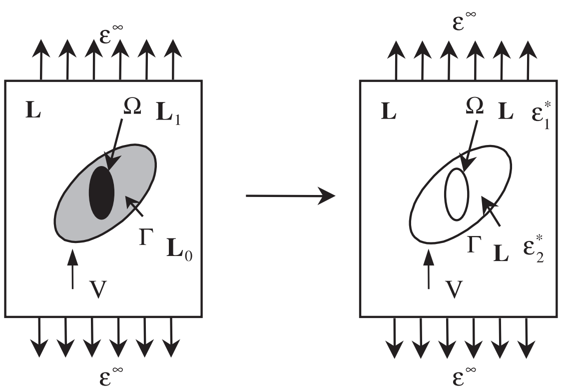

Fig. 1. Scheme of the double-inclusion method.

These two consistency equations allow one to determine the average eigenstrains $\boldsymbol{\varepsilon}_1^*$, $\boldsymbol{\varepsilon}_2^*$, and then the average stress and strain in the double-cell.

These two equations can be re-arranged into

$$\begin{aligned}&[(\boldsymbol{L}-\boldsymbol{L}_1)\boldsymbol{S}^V-\boldsymbol{L}]\boldsymbol{\varepsilon}_2^*+[(\boldsymbol{L}-\boldsymbol{L}_1)\boldsymbol{S}^\Omega-\boldsymbol{L}](\boldsymbol{\varepsilon}_1^*-\boldsymbol{\varepsilon}_2^*)=(\boldsymbol{L}_1-\boldsymbol{L})\boldsymbol{\varepsilon}^\infty,\\&[(\boldsymbol{L}-\boldsymbol{L}_0)\boldsymbol{S}^V-\boldsymbol{L}]\boldsymbol{\varepsilon}_2^*+\frac{c_1}{1-c_1}(\boldsymbol{L}-\boldsymbol{L}_0)(\boldsymbol{S}^V-\boldsymbol{S}^\Omega)(\boldsymbol{\varepsilon}_1^*-\boldsymbol{\varepsilon}_2^*)=(\boldsymbol{L}_0-\boldsymbol{L})\boldsymbol{\varepsilon}^\infty.\end{aligned}\tag{2}$$

This leads to the solution

$$\boldsymbol{\varepsilon}_2^*=\boldsymbol{A}_2\boldsymbol{\varepsilon}^\infty,\qquad \boldsymbol{\varepsilon}_1^*-\boldsymbol{\varepsilon}_2^*=\boldsymbol{A}_1\boldsymbol{\varepsilon}^\infty,\tag{3}$$

where $\boldsymbol{A}_2$ and $(\boldsymbol{A}_1+\boldsymbol{A}_2)$ represent the eigenstrain concentrations of the matrix and the inclusion, respectively. Their general forms can be easily deduced from (2), but it is better to write their specific forms after a choice of $\boldsymbol{L}$, and the shape and orientation of the double-cell ($\boldsymbol{S}^V$), have been decided for a specific objective.

For a two-phase composite, where the inclusions may be aligned or randomly oriented in a plane or in the space, and each double-cell may be aligned or take the same shape and orientation as the enclosed inclusion, the average stress and strain over all double-cells can be written as

$$\begin{aligned}&\bar{\boldsymbol{\sigma}}=\langle\boldsymbol{L}+\boldsymbol{L}[(\boldsymbol{S}^V-\boldsymbol{I})(\boldsymbol{A}_2+c_1\boldsymbol{A}_1)]\rangle\boldsymbol{\varepsilon}^\infty,\\&\bar{\boldsymbol{\varepsilon}}=\langle\boldsymbol{I}+\boldsymbol{S}^V(\boldsymbol{A}_2+c_1\boldsymbol{A}_1)\rangle\boldsymbol{\varepsilon}^\infty,\end{aligned}\tag{4}$$

where $\langle\cdot\rangle$ stands for the orientational average of the said quantity.

Following the idea of the double-inclusion model, the coefficient between the average stress and strain of the double-cell gives the effective moduli of the composite. After eliminating $\boldsymbol{\varepsilon}^\infty$ in Eq. (4) one arrives at

$$\boldsymbol{L}_\mathrm{c}=\big\langle\boldsymbol{L}+\boldsymbol{L}\big[(\boldsymbol{S}^V-I)(\boldsymbol{A}_2+c_1\boldsymbol{A}_1)\big]\big\rangle\big\langle\boldsymbol{I}+\boldsymbol{S}^V(\boldsymbol{A}_2+c_1\boldsymbol{A}_1)\big\rangle^{-1}\tag{5}$$

for the effective moduli tensor. Apparently $\boldsymbol{L}_\mathrm{c}$ depends on the choice of $\boldsymbol{L}$ and $\boldsymbol{S}^V$.

As pointed out by Hori and Nemat-Nasser (1993), the elastic moduli tensor $\boldsymbol{L}$ of the infinitely extended homogeneous medium can be taken as $\boldsymbol{L}_0$ or $\boldsymbol{L}_1$, or $\boldsymbol{L}_\mathrm{c}$ of the composite as one wishes, and this would lead to various estimates for the effective moduli. In what follows, we will demonstrate that, by judiciously choosing the shape and the orientation of the double-cell and the elastic moduli of the infinitely extended medium, the three micromechanical models cited above can be found. For clarity, we shall only consider the case that both tensors $\boldsymbol{L}_1$ and $\boldsymbol{L}_0$ are isotropic.

3. The Ponte Castaneda and Willis model

Ponte Castaneda and Willis (1995) proposed a rather novel approach to separate the spatial distribution of inclusions from the inclusion shape. It was based on the Hashin and Shtrikman (1962, 1963) variational principles originally developed in Willis (1977, 1980). In this development, the inclusions are taken to be ellipsoidal, and the distribution of inclusions is also taken to be ellipsoidal. Unlike in Willis (1977), these two ellipsoids do not have to be aligned and identically shaped. The distribution ellipsoid is defined from the conditional probability density function that represents the probability density for finding an inclusion centered at one point given that there is an inclusion centered at another point. The outcome of the development is that the effective moduli tensor of the composite is dependent on two ellipsoids, one characterizing the inclusion shape and the other characterizing the distribution function, and the results are explicit.

Now, consider the case that $\boldsymbol{L} = \boldsymbol{L}_0$, and the orientation of the double-cell is fixed but those of the ellipsoidal inclusions are randomly oriented. In this case, we have

$$\boldsymbol{A}_2 = 0, \qquad \boldsymbol{A}_1 = -\left[\boldsymbol{S}^{\Omega} - (\boldsymbol{L}_0 - \boldsymbol{L}_1)^{-1}\boldsymbol{L}_0\right]^{-1}, \tag{6}$$

from the average consistency conditions. It follows from (5) that the effective modulus tensor of the composite now writes as

$$\boldsymbol{L}_{\mathrm{c}} = \boldsymbol{L}_0\left\{\boldsymbol{I} - c_1\left[\langle \boldsymbol{A}_1\rangle^{-1} + c_1\langle \boldsymbol{S}^V \boldsymbol{A}_1\rangle\langle \boldsymbol{A}_1\rangle^{-1}\right]^{-1}\right\}. \tag{7}$$

Since the orientation of the double-cell is fixed, the Eshelby tensor $\boldsymbol{S}^V$ can be brought out of the orientational average to yield

$$\boldsymbol{L}_{\mathrm{c}} = \boldsymbol{L}_0\left\{\boldsymbol{I} - c_1\left[\langle \boldsymbol{A}_1\rangle^{-1} + c_1\boldsymbol{S}^V\right]^{-1}\right\}. \tag{8}$$

This result is exactly the same as the one derived by Ponte Castaneda and Willis (1995) (see also Hu and Weng, 2000). It is now clear that the shape of the double-cell in the double-inclusion model in fact characterizes the ellipsoidal distribution function of the inclusions. The corresponding microstructure is shown in Fig. 2. The orientation of the ellipsoidal double-cell must be fixed in order for the distribution function to

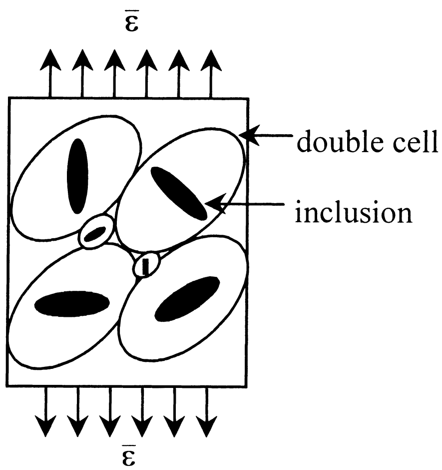

Fig. 2. The double-cell microstructure of the PCW model.

satisfy the symmetry requirement for the two-point joint probability distribution function. To obtain an isotropic composite, its shape must be spherical.

4. The Mori and Tanaka model

The MT model has been widely discussed in the literature (see, for instance, Weng 1984, 1990; Benveniste, 1987 for two distinct lines of approach). This model was not introduced with any known microstructure and it was not until Weng (1992) that a microstructure of Willis (1977) type was identified with it for the aligned ellipsoidal inclusions. For the randomly oriented ellipsoidal inclusions, it is still not known to possess any microstructure even though its moduli for a two-phase isotropic composite always remain inside the Hashin and Shtrikman (1963) and Walpole (1966) bounds. Thus any potential connection with the double-inclusion model will help reveal its possible microstructure, if it has any.

We now take $\boldsymbol{L} = \boldsymbol{L}_0$ but, unlike in PCW model, the shape and orientation of each double-cell is further taken to be identical to those of the inclusion enclosed in it. As the ellipsoidal inclusions are randomly oriented in space, the orientation of the double-cell must also change according to the orientation of each inclusion. That is, in the orientational average, the Eshelby tensor $\boldsymbol{S}^V$ can no longer be brought out of the averaging procedure in (5), but the fact that the double-cell has the same shape and orientation as the inclusion also implies that $\boldsymbol{S}^V = \boldsymbol{S}^\Omega$.

In this case, the eigenstrain concentration tensors carry the values

$$\boldsymbol{A}_2 = 0, \qquad \boldsymbol{A}_1 = \left[(\boldsymbol{L}_0 - \boldsymbol{L}_1)\boldsymbol{S}^\Omega - \boldsymbol{L}_0\right]^{-1}(\boldsymbol{L}_1 - \boldsymbol{L}_0) \tag{9}$$

and the average stress and strain of the inclusions and the matrix are

$$\boldsymbol{\sigma}_1 = \boldsymbol{L}_1\left[\boldsymbol{I} + \boldsymbol{S}^\Omega \boldsymbol{A}_1\right]\boldsymbol{\varepsilon}^\infty, \qquad \boldsymbol{\varepsilon}_1 = \left[\boldsymbol{I} + \boldsymbol{S}^\Omega \boldsymbol{A}_1\right]\boldsymbol{\varepsilon}^\infty, \tag{10}$$

$$\boldsymbol{\sigma}_0 = \boldsymbol{L}_0\boldsymbol{\varepsilon}^\infty, \qquad \boldsymbol{\varepsilon}_0 = \boldsymbol{\varepsilon}^\infty. \tag{11}$$

These results are immediately recognizable as the Eshelby result and, when used to form the overall stress and strain of the composite, they will lead to the MT model.

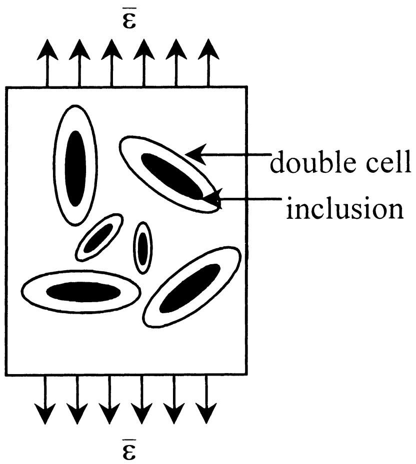

Fig. 3. The double-cell microstructure of the MT model.

Indeed after some algebra the effective moduli tensor $\boldsymbol{L}_c$ can be reduced from (5) to

$$\boldsymbol{L}_c = \boldsymbol{L}_0 \left\{ \boldsymbol{I} - c_1 [(1 - c_1)\langle \boldsymbol{A}_1 \rangle^{-1} + c_1 (\boldsymbol{I} - \boldsymbol{M}_0 \boldsymbol{L}_1)]^{-1} \right\}, \tag{12}$$

where $\boldsymbol{M}_0 = \boldsymbol{L}_0^{-1}$, the compliances tensor. This result is also the expression of the MT moduli (see, for instance, the expression given by Hu and Weng, 2000). The corresponding microstructure is sketched in Fig. 3. Since this type of microstructure leads to asymmetry of the two-point joint probability density function, this kind of distribution of the inclusions cannot be realized within the frame-work of microstructures defined by Ponte Castaneda and Willis (1995).

5. The Kuster and Toksoz model

Another explicit micromechanical model that is widely used in the geophysics community but remains little known in the mechanics community is the Kuster and Toksoz (1974) model. This model has recently been discussed in some detail by Berryman and Berge (1996), who also pointed out that this model was not known to have any realizable microstructure. In this section, we first follow Berryman and Berge's presentation of this model and then try to bring out its connection with the double-inclusion model. But before we do it we declare that the reference material of this model is also the matrix material, and its connection with the double-inclusion model is graphically shown in Fig. 4.

Now following Berryman and Berge (1996), the localization relation reads

$$\boldsymbol{\varepsilon}_i = \boldsymbol{G}_i \boldsymbol{\varepsilon}^{\infty}, \tag{13}$$

where $\boldsymbol{G}_i$ is the strain concentration tensor for the inclusion of type i, and the following relation also holds exactly for the orientational average,

$$(\boldsymbol{L}_c - \boldsymbol{L}_0)\bar{\boldsymbol{\varepsilon}} = \langle (\boldsymbol{L}_i - \boldsymbol{L}_0)\boldsymbol{G}_i \rangle \boldsymbol{\varepsilon}^{\infty}. \tag{14}$$

In this relation $\bar{\boldsymbol{\varepsilon}}$ is the composite strain, and $\boldsymbol{L}_c$ is the composite stiffness tensor. In order to evaluate $\bar{\boldsymbol{\varepsilon}}$ as a function of the applied strain $\boldsymbol{\varepsilon}^{\infty}$, Kuster and Toksoz (1974) proposed to place a composite inclusion with an ellipsoidal shape into the matrix material under the remote loading $\boldsymbol{\varepsilon}^{\infty}$ (see Fig. 4), and this yields

$$\bar{\boldsymbol{\varepsilon}} = \boldsymbol{G}_c \boldsymbol{\varepsilon}^{\infty}, \tag{15}$$

where $\boldsymbol{G}_c$ depends on the unknown $\boldsymbol{L}_c$ and also the shape of the composite inclusion. Then by means of Eshelby's solution, $\boldsymbol{G}_i$ and $\boldsymbol{G}_c$ are given by

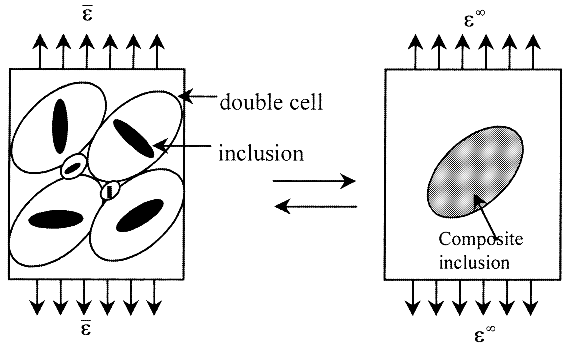

Fig. 4. Concept of the KT method.

$$\boldsymbol{G}_{\mathrm{i}} = (\boldsymbol{L}_0 - \boldsymbol{L}_{\mathrm{i}})^{-1}\boldsymbol{L}_0\boldsymbol{A}_{\mathrm{i}}, \qquad \boldsymbol{G}_{\mathrm{c}} = (\boldsymbol{L}_0 - \boldsymbol{L}_{\mathrm{c}})^{-1}\boldsymbol{L}_0\boldsymbol{A}_{\mathrm{c}} \tag{16}$$

with

$$\boldsymbol{A}_{\mathrm{i}} = -\left[\boldsymbol{S}_{\mathrm{i}} - (\boldsymbol{L}_0 - \boldsymbol{L}_{\mathrm{i}})^{-1}\boldsymbol{L}_0\right]^{-1}, \qquad \boldsymbol{A}_{\mathrm{c}} = -\left[\boldsymbol{S}^V - (\boldsymbol{L}_0 - \boldsymbol{L}_{\mathrm{c}})^{-1}\boldsymbol{L}_0\right]^{-1}. \tag{17}$$

In these expressions $\boldsymbol{S}^V$ is the Eshelby tensor characterizing the composite inclusion, and $\boldsymbol{S}_{\mathrm{i}}$ are those for the inclusion of type i. For the two-phase composite analyzed here with the inclusions having the same shape, we have

$$\boldsymbol{G}_{\mathrm{i}} = \boldsymbol{G}_1 = (\boldsymbol{L}_0 - \boldsymbol{L}_1)^{-1}\boldsymbol{L}_0\boldsymbol{A}_1 \tag{18}$$

in the local oriented coordinates.

Now with the expressions for $\boldsymbol{G}_{\mathrm{i}}$ and $\boldsymbol{G}_{\mathrm{c}}$, and noting that the orientational average operates only on $\boldsymbol{A}_1$, we arrive at

$$\boldsymbol{A}_{\mathrm{c}} = c_1\langle \boldsymbol{A}_1\rangle \quad \text{or} \quad -\left[\boldsymbol{S}^V - (\boldsymbol{L}_0 - \boldsymbol{L}_{\mathrm{c}})^{-1}\boldsymbol{L}_0\right]^{-1} = c_1\langle \boldsymbol{A}_1\rangle. \tag{19}$$

This leads to the stiffness tensor of the composite

$$\boldsymbol{L}_{\mathrm{c}} = \boldsymbol{L}_0\left\{\boldsymbol{I} - c_1\left[\langle \boldsymbol{A}_1\rangle^{-1} + c_1\boldsymbol{S}^V\right]^{-1}\right\}. \tag{20}$$

Interestingly, this result is exactly what we derived earlier for the PCW model. As a consequence the shape of the composite inclusion in the KT model also corresponds to the ellipsoidal region characterizing the distribution function of the inclusions in the PCW model. The KT model now has a realizable microstructure in the framework of PCW. For an isotropic composite, Kuster and Toksoz further assumed that $\boldsymbol{S}^{\mathrm{V}}$ be evaluated for a spherical inclusion (see Berryman and Berge, 1996 for more details), and thus the distribution function in the PCW model is spherical. But to make this connection safely, the volume concentration of the spheroidal inclusions and the aspect ratio w (the length-to-the-diameter ratio) must satisfy the PCW constraints $c_1 \leqslant w$ if $w \leqslant 1$, or $c_1 \leqslant (1/w)^2$ if $w \geqslant 1$. Otherwise the inclusions may be in contact or interpenetrate into each other and the microstructure cannot be constructed.

6. Some remarks for the choice $\boldsymbol{L} = \boldsymbol{L}_{\mathrm{c}}$

When the elastic moduli of the reference medium are chosen to be the (yet unknown) effective moduli of the composite, the double-inclusion configuration is exactly that of the generalized self-consistent model (Christensen and Lo, 1979) and the externally applied strain $\boldsymbol{\varepsilon}^{\infty}$ would be exactly equal to the weighted mean of the inclusion and matrix strains. (This latter point was also proved in Herve and Zaoui (1990), and independently confirmed by us.) In this case, one would normally anticipate that the result of the double-inclusion model be identical to that of the generalized self-consistent scheme, not the self-consistent scheme as concluded by Hori and Nemat-Nasser. Due to this unexpected outcome we decided to look into this special case again from the standpoint of stress and strain concentration tensors of the inclusion and the matrix.

We first note that the average stress and strain of the inclusion in the double-inclusion model are given by

$$\boldsymbol{\sigma}_1 = \boldsymbol{L}_1\left[\boldsymbol{I} + \boldsymbol{S}^{\Omega}\boldsymbol{A}_1 + \boldsymbol{S}^V\boldsymbol{A}_2\right]\boldsymbol{\varepsilon}^{\infty}, \qquad \boldsymbol{\varepsilon}_1 = \left[\boldsymbol{I} + \boldsymbol{S}^{\Omega}\boldsymbol{A}_1 + \boldsymbol{S}^V\boldsymbol{A}_2\right]\boldsymbol{\varepsilon}^{\infty} \tag{21}$$

and for the matrix

$$\begin{aligned}\boldsymbol{\sigma}_0 &= \boldsymbol{L}_0\left[\boldsymbol{I} + \boldsymbol{S}^V\boldsymbol{A}_2 + \frac{c_1}{1-c_1}(\boldsymbol{S}^V - \boldsymbol{S}^\Omega)\boldsymbol{A}_1\right]\boldsymbol{\varepsilon}^\infty,\\ \boldsymbol{\varepsilon}_0 &= \left[\boldsymbol{I} + \boldsymbol{S}^V\boldsymbol{A}_2 + \frac{c_1}{1-c_1}(\boldsymbol{S}^V - \boldsymbol{S}^\Omega)\boldsymbol{A}_1\right]\boldsymbol{\varepsilon}^\infty.\end{aligned} \tag{22}$$

When $\boldsymbol{L} = \boldsymbol{L}_c$ and $\boldsymbol{S}^V = \boldsymbol{S}^\Omega$, they further reduce to

$$\boldsymbol{\sigma}_1 = \boldsymbol{L}_1\left[\boldsymbol{I} + \boldsymbol{S}^\Omega(\boldsymbol{A}_1 + \boldsymbol{A}_2)\right]\boldsymbol{\varepsilon}^\infty, \qquad \boldsymbol{\varepsilon}_1 = \left[\boldsymbol{I} + \boldsymbol{S}^\Omega(\boldsymbol{A}_1 + \boldsymbol{A}_2)\right]\boldsymbol{\varepsilon}^\infty \tag{23}$$

and

$$\boldsymbol{\sigma}_0 = \boldsymbol{L}_0\left[\boldsymbol{I} + \boldsymbol{S}^\Omega\boldsymbol{A}_2\right]\boldsymbol{\varepsilon}^\infty, \qquad \boldsymbol{\varepsilon}_0 = \left[\boldsymbol{I} + \boldsymbol{S}^\Omega\boldsymbol{A}_2\right]\boldsymbol{\varepsilon}^\infty, \tag{24}$$

where

$$\boldsymbol{A}_1 + \boldsymbol{A}_2 = \left[(\boldsymbol{L}_c - \boldsymbol{L}_1)\boldsymbol{S}^\Omega - \boldsymbol{L}_c\right]^{-1}[(\boldsymbol{L}_1 - \boldsymbol{L}_c)] \tag{25}$$

$$\boldsymbol{A}_2 = \left[(\boldsymbol{L}_c - \boldsymbol{L}_0)\boldsymbol{S}^\Omega - \boldsymbol{L}_c\right]^{-1}[(\boldsymbol{L}_0 - \boldsymbol{L}_c)]. \tag{26}$$

These are exactly the Eshelby results when the inclusion and the matrix are respectively embedded in the effective composite. It is also the exact concept of the self-consistent method. Thus the conclusion reached here is the same as that of Hori and Nemat-Nasser; that is, the double-inclusion model delivers the self-consistent result, not the generalized self-consistent result.

There is a strong possibility that this unexpected outcome is due to the neglected term in Hori and Nemat-Nasser's generalization from the local eigen-field to the average eigen-field for each phase. They pointed out correctly that, when the eigenstrain $\boldsymbol{\varepsilon}_2^*$-field of the matrix is uniform, such a generalization is exact, but that, when this field is not uniform, its exact nature is of a less certainty. If $\boldsymbol{L}$ is chosen as $\boldsymbol{L}_0$ (such as in the choice leading to the PCW, MT, and KT models discussed earlier), this eigenstrain is zero and thus, the generalization is exact. But when $\boldsymbol{L}$ is chosen to be equal to $\boldsymbol{L}_c$ this field is not zero and not uniform, and, as such, the average self-consistency conditions given in Eq. (1) may not be exact. Thus this neglected term may have caused the double-inclusion model to lead to the self-consistent scheme instead of the generalized self-consistent scheme. But evaluation of this neglected term for a general ellipsoid is not a trivial matter and we were not able to reach a definitive conclusion for it.

7. Conclusions

The following new connections have been established in this paper:

(i) If the double-cell is fixed, the double-inclusion model leads to the same results as the PCW model. The shape of the double-cell also represents the distribution function of the inclusions in the PCW microstructure.

(ii) When the double-cells have the same shape and orientation of the enclosed inclusions, the double-inclusion model will produce the MT results.

(iii) The widely used KT model in the geophysics community gives the same predictions as the PCW method. As such, it now has a known microstructure that was previously considered to be non-existent, but in order for this model to stay within the PCW framework the range of applicability of the inclusion concentration must be tied with its aspect ratio as set out in the PCW model.

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