Elastic metamaterials with local resonances: an overview

Xiaoming Zhou, a) Xiaoning Liu, b) and Gengkai Hu c)

Key Laboratory of Dynamics and Control of Flight Vehicle, Ministry of Education, and School of Aerospace Engineering, Beijing Institute of Technology, Beijing 100081, China

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Abstract Metamaterials are artificial composite materials engineered to have properties that may not be found in nature. By exploring locally resonant effect of the building units, elastic metamaterials are able to possess negative values of effective mass, effective bulk or shear modulus. Mass-spring and continuum material versions of these elastic metamaterials are reported and the physical mechanisms of negative effective parameters are demonstrated. Applications of metamaterials to acoustic cloaking and superlensing are also discussed. © 2012 The Chinese Society of Theoretical and Applied Mechanics. [doi:10.1063/2.1204101]

Keywords metamaterials, negative mass, negative modulus, resonance, cloaking, superlensing

I. INTRODUCTION

Matter is made of atoms and molecules, their interactions under external stimuli, usually idealized by springs, give rise to resistances. In the continuum sense, these resistances are characterized by effective modulus or mass density. It is intuitive to define a positive modulus for a spring connecting two atoms elongating under a tensile force, and a positive inertia mass for an atom with the acceleration responding in phase to the applied force. Most natural materials behave in this manner. However with the development of composite material technology, we can design complex microscopic units so that the composite as a whole can respond to external stimuli in an unusual way. For example, a composite with coated particles may have a stopband under an harmonic wave loading, which can be interpreted by negative mass density, a) and wave characteristic of a material made of Helmholtz resonators can be explained by negative effective modulus. b) These unusual material properties recall serious efforts to elucidate the implied mechanism and to explore their impact on modern technology. This is also one of major efforts in the last decade in different disciplines, largely driven by the development of electromagnetic materials with negative permeability or permittivity. 3,4 Materials with properties not readily observed in nature are termed as metamaterials. For elastic metamaterials, they include composite materials with negative modulus, negative mass density or anisotropic mass. Therefore metamaterials enlarge significantly the choice for material selection with a targeted application and provide an unprecedented way for manipulating wave propagations.

The first acoustic metamaterial consists of lead particles coated with soft rubber layers and embedded in an epoxy matrix. 5 A stopband for acoustic wave has been attributed to negative effective mass of the composite. Fang et al. proposed the Helmholtz resonators to generate negative effective modulus. 6 Double negative index material has been designed with two populations of coated inclusions, and monopole and dipolar resonances are engineered to be present at the overlapped frequencies. 5,6 A more compact design of double negative index materials is recently proposed based on a chiral structure. 7 The rotation resonance of the particles is introduced to produce negative effective bulk modulus, and the translational resonance induces negative effective mass. Negative effective shear modulus is related to the quadruple resonance, and a planar metamaterial with a negative effective shear modulus is conceived by enhancing the quadruple resonance. 5,8,9 In order for illustrating the mechanism of local resonance and its effect on macroscopic property, mass-spring systems are usually used as model systems, e.g. the dipolar resonance of a coated inclusion can be well represented by a mass-spring resonator. 10,11 Many works have been devoted to examine the transmission property of mass-spring systems involving the local resonances. 12,13

Metamaterials with microstructures carefully patterned in the space are able to control acoustic wave for a targeted application. The relation between the function of a device and the necessary material spatial distribution can be established by the transformation method, 14-17 based on the form invariance of Helmholtz equation for acoustic waves. For elastic waves, the Navier’s equation does not retain the form invariance under a general spatial mapping. 18 However it has been shown that elastic waves can still be controlled approximately by patterning the microstructure of composites under appropriate assumptions. 19-21

In this review, we will first focus on clarifying the mechanisms of the unusual properties through simple mass-spring systems. The mechanism of negative effective mass and modulus, and their interplay with macroscopic property will be explored. The continuum versions of metamaterials will then be examined and their effective material properties will be derived from homogenization techniques. Finally, some applications of elastic metamaterials will be presented in Section III.
II. MECHANISMS OF ELASTIC METAMATERIALS

A. Discrete mass-spring system

1. Negative effective mass

The mechanism of negative effective mass can be discovered by the mass-spring structure shown in Fig. 1, where a hollow sphere of the weight \( M_0 \) connects a rigid sphere of weight \( m \) by two massless and elastic springs with equal spring coefficient \( G \). In the time harmonic case, the equilibrium equation of the inner mass \( m \) results in

\[
\frac{x}{X} = \frac{\omega_1^2}{\omega_1^2 - \omega_0^2},
\]

where \( x \) and \( X \) are respectively the displacement of the inner and outer mass, and \( \omega_1 = \sqrt{2G/m} \) is the resonant frequency. The composite hollow structure can be considered as an equivalent solid object and the effective mass \( M_{\text{eff}} \) of the unit can be defined as the total momentum divided by the velocity of the outer mass\(^{11}\)

\[
M_{\text{eff}} = M_0 + \frac{m\omega_1^2}{\omega_1^2 - \omega_0^2}.
\]

Due to the negative momentum of the inner mass overwhelming the system, the effective mass \( M_{\text{eff}} \) takes negative values at frequencies ranging from \( \omega_0 \) to \( \omega_0 \sqrt{(M_0 + m)/M_0} \). When the inner mass is fixed, effective dynamic mass in Eq. (2) becomes\(^{22}\)

\[
M_{\text{eff}} = M_0 \left(1 - \frac{\omega_0^2}{\omega_1^2}\right).
\]

In such case, effective mass is negative below a cut-off frequency \( \omega_0 = \sqrt{2G/M_0} \).

When the mass-spring structures are connected by springs of elastic coefficient \( K \), the dispersion relation for an infinite lattice system with the lattice distance \( a \) is derived to be\(^{12}\)

\[
M_{\text{eff}}\omega^2 = 4K\sin^2\frac{qa}{2},
\]

where \( q \) is the Bloch wave vector. From Eq. (4), it is seen that the wave vector \( q \) takes imaginary values at frequencies of negative effective mass. It means that the lattice waves will be forbidden in this frequency band. To verify the blocking effect due to negative effective mass, a finite lattice system is considered and shown in Fig. 1(b). The blocking effects at the frequencies of negative effective mass can be evaluated by the transmission coefficient \( T = |X_N/X_0| \). Figure 2 shows an experimental setup constructed to measure the transmission spectrum. For a lattice system comprising seven units, Figs. 3(a) and 3(b) show theoretical and experimental results of the transmission in the case of free and fixed inner mass.\(^{12,22}\) It is found that the transmissions are greatly lowered at the frequencies of negative effective mass both for free and fixed inner mass, conforming the negative effective mass exhibited by the mass-spring structure.

Anisotropic effective mass can be realized by a different set of springs placed vertically between the inner and outer mass in the structure of Fig. 1(a). In this case, the mass density is a tensor value instead of a scalar for conventional materials.

2. Negative effective modulus

Negative effective modulus of elastic metamaterials can be produced by the local resonant effect, as firstly demonstrated in a hollow waveguide attached by an array of sub-wavelength Helmholtz resonators.\(^2\) The mechanism of negative modulus will be discovered by a mass-spring model given below. By inducing the monopolar resonance of bubble-contained-water spheres in an epoxy host, Ding et al. also propose a composite with a negative effective bulk modulus,\(^6\) which facilitates the fabrication of acoustic left-handed metamaterials.\(^5\) Wu et al. design a fiber composite with a negative effective shear modulus by introducing the quadrupolar resonances.\(^8\)^\(^9\)

To understand the underlying physics of negative effective modulus, the mass-spring structure that exhibits negative effective stiffness is shown in Fig. 4(a), where three massless springs with stiffness \( k \) and \( K \) are serially connected, and a rigid sphere of weight \( m \) is attached to the middle spring by two massless rigid bars with an angle \( \alpha \). All joints are free of friction and the springs are confined on the \( x \) axis. Consider that a time harmonic force \( F(t) \) is applied to the structure. \( x, \bar{x} \) and \( y \) denote respectively the offset of the end, middle joint and mass with respect to the equilibrium position.

According to the Newton’s law of motion, the following relations can be obtained

\[
m\frac{d^2y}{dt^2} = f, \quad f = 2(F - 2K\bar{x})\tan \alpha, \quad F = k(x - \bar{x}).
\]

For an infinitesimal disturbance, there exists the geometrical relation \( \bar{x} = y \tan \alpha \). Effective stiffness is de-
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2. Fig. 2. The experimental setup constructed to measure the transmission of a finite lattice system.

3. Fig. 3. Transmission coefficients given by theory and experiment in the case of the free inner mass (a) and the fixed inner mass (b).

4. Fig. 4. (a) The mass-spring structure with negative effective stiffness; (b) Effective stiffness as function of the frequency.

5. Fined to be \( \tilde{K} = \bar{F}/(2\dot{u}) \) and

\[
\tilde{K} = \frac{1}{2} \left[ \frac{(m\omega^2 - 4K\tan^2\alpha)k}{m\omega^2 - (4K + 2k)\tan^2\alpha} \right].
\]

Equation (6) reveals that the effective stiffness can be negative within the frequency range from \( \tan\alpha\sqrt{4K/m} \) to \( \tan\alpha\sqrt{(4K + 2k)/m} \), and \( \omega_0 = \tan\alpha\sqrt{(4K + 2k)/m} \) is the resonant frequency. When the frequency approaches zero, \( \tilde{K} \) reduces to the static stiffness \( K_S = Kk/(2K + k) \). A typical variation of the effective stiffness as a function of frequency is shown in Fig. 4(b), where the parameter \( k = 2K \) is taken. The physical mechanism of negative effective modulus is that the oscillation of the sphere in the vertical direction induces an inertial force in the horizontal direction, and the resonance of the sphere will lead to an expansion of the structure, while loaded under a compressive force.

3. Double-negative system

A material with simultaneous negative mass and modulus has a negative refractive index, due to anti-
parallel nature of phase and group velocities. This double negative material is also called acoustic left-handed material, or double negative index material, first systematically examined by Veselago for its electromagnetic counterpart. The mass-spring structures analyzed above can be combined together to produce an elementary unit of a double negative material, as shown in Fig. 5(a). According to the Bloch’s theorem, the dispersion relation of an infinite lattice system comprising such a unit cell can be solved from a forth-order eigen-equation. For the parameters \( k_1 = 800, k_2 = 400, k_3 = 250, m_1 = 1, m_2 = m_3 = 10, \alpha = 10^\circ \), the dispersion curves of the periodic system with negative mass, negative modulus, and double negative parameters are shown in Fig. 5(b). The full band gap exists in the frequency region \( 5 < \omega < 10 \) for either negative mass or modulus. However in the double negative system, a passband \( (8.3 < \omega < 10) \) is obviously emerged out of the bandgap, allowing wave transmission but with a negative phase velocity.

**B. Continuum material system: composites with coated spheres**

By properly exploring the dipolar, monopolar and quadrupolar resonances in composites with coated sphere embedded in a solid matrix, negative effective mass, bulk and shear modulus of the composites can be designed. To this end, we have to first develop a homogenization theory which can estimate the dynamic effective property of a composite under a wave loading. An analytic model made of a doubly coated sphere in the case of compressional wave incident on the coated sphere has been analyzed, as shown in Fig. 6. The averaged fields of displacements, bulk strains, maximum shear strains for each phase of the doubly coated sphere can be computed. Locally resonant effects of the building unit that accounts for negative effective material parameters can be discovered by analyzing these averaged fields. Negative effective mass arises from the dipolar resonance in composites with rubber-coated lead spheres suspended in the epoxy matrix. Negative effective bulk modulus is realized due to the monopolar resonance of bubble-contained-water spheres in an epoxy host. Negative effective shear modulus is designed when particles undergo the quadrupolar resonance. The detailed discussions will be given in the following.

**1. Negative effective mass**

For a plane harmonic wave, the equation of motion for an elastic material is written as

\[
\nabla \cdot \mathbf{\sigma} = -i \omega \mathbf{p},
\]

Integrating Eq. (7) in a sphere with the volume \( V \), the outer surface \( S \) and the radius \( r \), we can define the macroscopic equation of motion

\[
\int_S \mathbf{d} s \cdot \mathbf{\sigma} = -i \omega V \langle \mathbf{p} \rangle,
\]

where we have used the Green formula and the averaging momentum \( \langle \mathbf{p} \rangle \) is defined as \( \langle \mathbf{p} \rangle = \int_V \mathbf{p} \, dV/V \). The integration in the left-hand side of Eq. (8) stands for the total force \( \mathbf{F} = \int_S \mathbf{d} s \cdot \mathbf{\sigma} \) acting on the spherical surface \( S \). From Eq. (8), the macroscopic equations of motion for each region of a doubly coated sphere are written as

\[
\int_{S_1} \mathbf{d} s \cdot \mathbf{\sigma} = -i \omega V_1 \langle \mathbf{p} \rangle_1,
\]

\[
\int_{S_2} \mathbf{d} s \cdot \mathbf{\sigma} = -i \omega V_2 \langle \mathbf{p} \rangle_2,
\]

\[
\int_{S_3} \mathbf{d} s \cdot \mathbf{\sigma} = -i \omega V_3 \langle \mathbf{p} \rangle_3.
\]
\[ \iint_{S_2} d\mathbf{s} \cdot \mathbf{\dot{\sigma}} - \iint_{S_1} d\mathbf{s} \cdot \mathbf{\dot{\sigma}} = -i\omega V_2 \langle p \rangle_2, \]  
\[ \iint_{S_3} d\mathbf{s} \cdot \mathbf{\dot{\sigma}} - \iint_{S_2} d\mathbf{s} \cdot \mathbf{\dot{\sigma}} = -i\omega V_3 \langle p \rangle_3, \]  
where \( S_1, S_2 \) and \( S_3 \) represent separately the sphere-coating interface \( r = r_1 \), the coating-matrix interface \( r = r_2 \), and the external surface \( r = r_3 \). \( V_i \) denotes the volume occupied by the \( i \)th region. Since the velocity of the composite sphere that we observe is the velocity of the host material, we define the velocity of the composite as \( \langle \mathbf{\dot{u}} \rangle_{\text{total}} = \phi_3 \langle \mathbf{\dot{u}} \rangle_3 \). The total momentum of the composite is the sum of momentum of each region. Then the dynamic effective mass density of the composite is defined as

\[ \rho_{\text{eff}} = \frac{1}{4\pi r^3/3} \langle \mathbf{p} \rangle_{\text{total}} = \frac{\phi_1 \langle p \rangle_1 + \phi_2 \langle p \rangle_2 + \phi_3 \langle p \rangle_3}{\langle p \rangle_3}. \]  

Consider the composite consisting of rubber-coated lead spheres embedded in an epoxy matrix. Figures 7(a) and 7(b) show effective mass density normalized to the static one \( \rho_{\text{eff}}/\rho_s \) and the momentum ratios \( p_1/p_3 \) and \( p_2/p_3 \) between constituents, where the subscript 1, 2, 3 represent respectively the particle, the coating and the matrix. It can be seen that there are two resonant negative-mass bands, with the lower and higher frequency bands induced respectively by the out-of-phase motions of the lead sphere and rubber coating with respect to the epoxy matrix, as shown schematically in Figs. 7(c) and 7(d).

The physics disclosed here coincides with that discovered by the mass-spring structure shown in Fig. 1(a). In this example, negative effective material parameters are available in a limited bandwidth due to the resonant effect. To broaden the frequency range, Lee et al. found that a membrane with a fixed boundary can be considered as a metamaterial with negative effective mass below a cut-off frequency. Based on the mass-spring model of Fig. 1 with the fixed inner mass, Yao et al. developed the continuum model, which is very promising for low frequency noise isolation.

2. Negative effective bulk modulus

The constitutive relation for the \( i \)th region of the doubly coated sphere is given by

\[ \langle \mathbf{\dot{\sigma}} \rangle_i = 3\lambda_i \langle \varepsilon_b \rangle_i \mathbf{I} + 2\mu_i \langle \mathbf{\varepsilon} \rangle_i, \]  
where the averaging field \( \langle \mathbf{\dot{F}} \rangle_i \) is defined as \( \langle \mathbf{\dot{F}} \rangle_i = (1/V_i) \int_{V_i} \mathbf{\dot{F}} dV, \varepsilon_b = \text{tr}\varepsilon/3 \) is the bulk strain and \( \mathbf{I} \) is the second-order unit tensor. The strain tensor \( \varepsilon \) is related to the displacement field \( \mathbf{u} \) by

\[ \varepsilon = \frac{1}{3} \nabla \mathbf{u}, \]  

Fig. 7. (a) Effective mass density, (b) the ratio of averaged momentums, and the schematic view of the displacement for a doubly-coated sphere at frequencies associated with (c) the first and (d) the second negative-mass bands.\(^{24}\)

\[ \varepsilon = \frac{1}{2} (\nabla \mathbf{u} + \mathbf{u} \nabla). \]  

The averaging bulk stress for the \( i \)th region \( \langle \sigma_b \rangle_i = \langle \text{tr}\mathbf{\dot{\sigma}} \rangle_i / 3 \) can be calculated by

\[ \langle \sigma_b \rangle_i = 3\kappa_i \langle \varepsilon_b \rangle_i, \]  
where \( \kappa_i = \lambda_i + 2\mu_i/3 \) is the bulk modulus. The effective bulk modulus of the composite can be defined as

\[ \kappa_{\text{eff}} = \frac{3\langle \sigma_b \rangle_{\text{total}}}{\langle \varepsilon_b \rangle_{\text{total}}} = \frac{\kappa_1 \phi_1 \langle \varepsilon_b \rangle_1 + \kappa_2 \phi_2 \langle \varepsilon_b \rangle_2 + \kappa_3 \phi_3 \langle \varepsilon_b \rangle_3}{\phi_1 \langle \varepsilon_b \rangle_1 + \phi_2 \langle \varepsilon_b \rangle_2 + \phi_3 \langle \varepsilon_b \rangle_3}. \]  

For the composite of bubble-contained-water spheres embedded in an epoxy matrix. Figures 8(a) and 8(b) give effective bulk modulus normalized to the static one \( \kappa_{\text{eff}}/\kappa_{\text{eff}}^s \), and the averaged bulk strains \( \varepsilon_i = \varphi_i \langle \varepsilon_b \rangle_i \) of each constituents versus the total bulk strain \( \varepsilon_{\text{total}} = \sum \varepsilon_i \). In Fig. 8(a), the composite exhibits...
3. Negative shear modulus

Go back to Eq. (11) and consider the deviatoric part of the averaging strain \( \langle \varepsilon'' \rangle_i \), it can be expressed as

\[
\langle \varepsilon'' \rangle_i = \varepsilon''_i = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.
\]  

From the constitutive equation, the averaging deviatoric stress \( \langle \sigma'' \rangle_i \) is related to the deviatoric strain through the shear modulus of the \( i \)th region

\[
\langle \sigma'' \rangle_i = 2\mu_i \langle \varepsilon'' \rangle_i,
\]

or equivalently,

\[
\langle \tau \rangle_i = 2\mu_i \langle e \rangle_i,
\]

where the averaging shear strain \( \langle e \rangle_i \) is defined as

\[
\langle e \rangle_i = \frac{1}{2} [2\varepsilon''_i - (-\varepsilon''_i)] = \frac{3}{2}\varepsilon''_i,
\]

and \( \langle \tau \rangle_i \) is the corresponding averaging shear stress in the \( i \)th region. The effective shear modulus of the composite can be defined as

\[
\mu_{\text{eff}} = \frac{\langle \tau \rangle_{\text{total}}}{2 \langle e \rangle_{\text{total}}} = \mu_1 \phi_1 \langle e \rangle_1 + \mu_2 \phi_2 \langle e \rangle_2 + \mu_3 \phi_3 \langle e \rangle_3 \]

\[
\frac{\phi_1 \langle e \rangle_1 + \phi_2 \langle e \rangle_2 + \phi_3 \langle e \rangle_3}{\phi_1 + \phi_2 + \phi_3}.
\]

Consider rubber-coated epoxy spheres embedded in a matrix of Polyethylene foam. Effective shear modulus of the composite is shown in Fig. 9(a). The method predicts negative effective shear modulus in two narrow frequency bands. To understand the mechanism of negative effective shear modulus, Fig. 9(b) plots the averaged shear strains \( e_i = \phi_i \langle e \rangle_i \) of each constituent versus the total shear strain \( e_{\text{total}} = \sum e_i \) as a function of frequency. The averaged shear strain of the matrix could become negative, in correspondence with the negative effective shear modulus. We simply assume that a sphere will become a prolate or oblate spheroid of the constant volume by the shear deformation. Figure 9(c) shows a schematic picture of the deformation profile of a doubly coated sphere at frequency of negative effective shear modulus. When the composite sphere (the outermost surface) is deformed with the prolate shape, the coating material also has a prolate shape but with a larger aspect ratio, as seen in Fig. 9(b). The inner core has no shear deformation at any frequency. So the matrix cover is actually compressed in the \( x-y \) plane, macroscopically behaving as the oblate-shape deformation. Since the Polyethylene foam is stiffer than the soft rubber, the composite sphere will be under the same loadings to the Polyethylene foam, denoted by the arrows in Fig. 9(c). The out-of-phase phenomenon between the deformation and the applied stress is the origin of negative effective shear modulus.
Recently, Wu et al. designed elastic metamaterials with negative shear modulus based on such a mechanism.\cite{wu2012}

The above numerical results demonstrate that the out-of-phase effect induced by resonance is the origin of negative effective material parameters. With help of the averaged physical fields, the analytic model provides a unified explanation for this unusual phenomenon. It is found that the first three scattering channels correspond respectively to the rigid-body movement \((n = 1)\), the volume (bulk) deformation \((n = 0)\), and the axisymmetric (shear) deformation of a constant volume \((n = 2)\). Negative effective mass is induced by negative total momentum of the composite for a positive momentum excitation. Negative effective bulk modulus appears for composites with an increasing (decreasing) total volume under a compressive (tensile) stress. Negative effective shear modulus describes composites with axisymmetric deformation under an opposite axisymmetric loading.

### 4. Double negative system

Although by adjusting the material parameters, the composites with coated particles can independently realize the negative mass density, bulk and shear modulus, but it is very difficult to overlap the frequency to form a double negative system. So another mechanism should be explored, chiral elastic metamaterials could exhibit negative effective bulk modulus\cite{wu2012,li2012} with help of the rotational resonance. The unit cell of the chiral metamaterial is shown in Fig. 10(a) and composed of a heavy cylindrical core with soft coating embedded in a matrix, and a number of slots are cut out from the coating material. The slots are equi-spaced in azimuth and oriented at an angle with respect to the radial direction.

Due to the fact that the slots in the coating are not oriented towards the center, the unit cell lacks in any planes of mirror symmetry, leading to the macroscopic chirality of metamaterials.\cite{wu2012,li2012} By introducing chiral microstructure, the rotational resonance of the particle may be coupled to the overall dilatation of the unit cell, and result in negative effective bulk modulus. The mass-spring model given in Fig. 10(b) explains negative effective modulus due to the rotational resonance. For this chiral metamaterial, negative effective mass can still be produced with help of the translation resonance of the core, so we can overlap the frequency of the translation and the rotation resonances by adjusting the angle of the slots, their spacing to form a metamaterial with double negative material parameters.\cite{wu2012}

### III. APPLICATIONS OF ELASTIC METAMATERIALS

#### A. Cloaking in quasi-static approximation

The cancellation mechanism for electromagnetic waves was proposed by Ali and Engheta,\cite{ali2008} who utilized a plasmonic or metamaterial coating to cover a spherical or cylindrical dielectric core. By adjusting the material and geometrical parameters, they found that at certain configurations, the total scattering cross section of this coated sphere can be extremely lowered. Zhou and Hu have extended the transparency phenomenon to multi-layered sphere, coated spheroids and two-phase composites by introducing the “neutral inclusion” concept.\cite{zhou2009} Afterwards the idea is further extended to the regime of acoustic and elastic waves.\cite{wu2012,li2012} Consider an inclusion of random shape is put into an infinite matrix. The inclusion is characterized by the bulk modulus \(\kappa_s\), shear modulus \(\mu_s\), mass density \(\rho_s\), and the infinite matrix has the material parameters \(\kappa_0\), \(\mu_0\), and \(\rho_0\). The inclusion can be made of either a homogeneous medium or a heterogeneous material. For the latter, \(\kappa_s\), \(\mu_s\), and \(\rho_s\) then denote the effective material parameters of the heterogeneous material (i.e. the inclusion). When the material properties of the inclusion are the same as those of the background medium, the wave fields outside of this inclusion will not be disturbed. In other
word, the inclusion will not be “seen” by an outside observer and become undetectable. This is the basic idea of the “neutral inclusion” concept. When the region is made of a homogeneous material, this is a trivial case. However, if the region is made of a heterogeneous material, there are many design possibilities for equating its effective material property to that of the background medium. According to the neutral inclusion concept, the key point to achieve transparency is to determine the effective parameters of the (heterogeneous) inclusion, or to find an equivalent homogeneous medium for the inclusion.

In the quasi-static approximation, heterogeneous materials can be described by equivalent homogeneous ones based on the homogenization technique. In this approximation, a neutral inclusion can be a simple pattern (coated sphere, coated spheroid, etc). When a neutral inclusion is embedded in a material made of assemblages of such pattern with gradual sizes (in order to fill the whole space), it will not perturb the static stress fields outside of this inclusion. Although the neutral inclusion is defined in the quasi-static case, it can still help to realize transparency in the full-wave scattering case when the wavelength becomes comparable to the size of the inclusion. For elastic wave transparency of a solid sphere coated with metamaterials, a coated sphere assemblage should be considered and their effective parameters can be evaluated by the Hashin-Shtrikman (HS) bound. By equating effective parameters to those of the background material, transparency conditions can be obtained. As an example, a fluid cover can be designed to cloak an aluminum sphere immersed in water. Figures 11(a) and 11(b) present the near-field contour plots of the radial component of the scattered displacement fields for an uncoated aluminum sphere and that with an optimized cloak, respectively. It can be seen that the uncoated sphere produces strong and nonuniform scattering field in the matrix, especially in the region adjacent to the sphere. However, when the cloaking material covers the sphere, the scattering is dramatically reduced whilst the field strength within the cloak is large. In the both cases, the displacement field inside the aluminum sphere is negligibly small due to its large modulus. It is thus demonstrated that, with the cloaking material, the impenetrable sphere can indeed achieve acoustic transparency. This property of the composite system can lead to potential applications in underwater stealth technology.

B. Transformation acoustics

When microstructures with varied geometric parameters are patterned in the space, metamaterials then have gradient effective material parameters in the space, and they can be used to control the propagation of acoustic and elastic waves. The relation correlating functional metamaterials to the necessary material spatial pattern have been established by Greenleaf et al., Pendry et al., and Leonhardt. The basic idea is based on the form-invariance of governing equations under a general coordinate transformation, and such a topological effect in spatial mapping can be mimicked by spatial distribution of materials. For acoustic waves, the Helmholtz equation is written as

$$\left(\rho_{ij}^{-1} p_{,i}\right)_j = -\frac{\omega^2}{\kappa} p,$$

where $p$ is the sound pressure, $\kappa$ is the bulk modulus, $\rho_{ij}$ is the mass tensor. Consider a mapping from an initial space to a curvilinear space $x \rightarrow x'$. In the new
space \((x')\), Eq. (20) becomes
\[
A_j' (\rho_{ij}^{-1} A_i' p_{ij})_{j'} = -\frac{\omega^2}{\kappa} p,
\]
where \(A_j' = \partial x_j'/\partial x_i\) is the Jacobin matrix describing the coordinate transformation from \(x\) to \(x'\). Introduce the Jacobian \(J = \det(A)\) and consider the relation \((J^{-1} A_j')_{j'} = 0^3,\) Eq. (21) is further expressed as
\[
(\rho' \delta_{ij} p_{ij})_{j'} = -\frac{\omega^2}{\kappa'} p,
\]
with
\[
\rho'^{-1} J = J^{-1} A_i' A_j' \rho_j^{-1}, \quad \kappa' = J \kappa.
\]  

It is found that Eq. (22) has the same form as Eq. (20), meaning that the topological change under the mapping can be mimicked by the changed material parameters \(\rho' \delta_{ij}\) and \(\kappa'\). From Eq. (23), material parameters can be uniquely determined for a specific space mapping. As an example, the space mapping for a cylindrical cloak that guides waves around an object is written as
\[
r' = a + r(b - a)/b, \quad \theta' = \theta,
\]
where \(r\) and \(\theta\) are polar coordinates, and \(a\) and \(b\) are respectively the inner and outer radii of the cloak. From Eq. (25), the mass density and modulus of the acoustic cloak are
\[
\rho' = \rho \frac{r'}{r' - a}, \quad \rho'_\theta = \rho \frac{r'}{r'} - a,
\]
\[
\kappa' = \kappa \left(\frac{b - a}{b}\right)^2 \frac{r'}{r' - a}.
\]

Figure 12 shows the contour plot of pressure fields of a plane wave incident leftwards on a rigid cylinder without and with the cloak. It is seen that a shadow is left behind the uncloaked cylinder (Fig. 12(a)), while the cloak could guide the incoming wave around the object without any scatterings, as if there is nothing there (Fig. 12(b)). The distributions of mass density and bulk modulus are shown in Fig. 13. It is found that anisotropic mass density is necessary and very difficult to be realized by conventional materials. Metamaterials with designable material parameters offer us a good candidate serving for transformation materials. A prototype of acoustic cloak made of Helmholtz resonators has been designed and validated by experiments.\(^\text{38}\)

Under a general coordinate transformation, the elastodynamic equation (Navier’s equation) in linear-elastic dynamics will be transformed into the Willis’ one.\(^\text{18}\) Otherwise the form invariance is maintained only when the stiffness tensor \(C_{ijkl}\) losses the minor symmetry,\(^\text{39}\) and leading to the asymmetric transformation relation. It is necessary to find a symmetric transformation relation to facilitate the design of transformation elastic materials. Regarding to this issue, the deformation-based transformation method is suggested by Hu et al.\(^\text{40}\) In the local principal coordinate, the governing equation in the transformed space is
\[
\frac{\partial \sigma_{ij}'}{\partial x_j'} = \rho' \frac{\partial^2 u_i'}{\partial t^2}, \quad \sigma'_{ij} = C'_{ijkl} \frac{\partial u_k'}{\partial x_l'},
\]

The transformation relations are given by
\[
C'_{ijkl} = \frac{1}{J} A_i' A_j' A_k' A_l' C_{ijkl},
\]
\[
\rho' = \frac{1}{J} A_i' A_j' \delta_{ij} \rho.
\]

Figure 14 shows an example of a rotator for shear waves designed with help of Eq. (27). Note that transformation materials with the parameters (27) are within the framework of linear-elastic dynamics and potentially realizable by metamaterial technology. To derive Eq. (27), the local affine transformation is assumed to retain locally the form invariance of the Navier’s equation. Such approximation is pertinent when the deformation gradient is small or the size of the transformation material is sufficiently larger than the operating wavelength. It is also shown that the transformation relations Eq. (27) keep the eikonal equation invariant, i.e. the wave path can be controlled exactly, but leave the amplitude of the wave to be controlled approximately.\(^\text{19}\) Based on the theory of transformation elasticity, elastic metamaterials can be designed to control elastic waves at will, and many engineering applications could be anticipated.
C. Acoustic super-resolution imaging

Metamaterials could serve as acoustic lenses that create images with the spatial resolution beyond the diffraction limit. Acoustic images are obtained from the scattering fields of the objects under wave excitations. Among the scattered fields, evanescent wave components with large spatial frequency carry the subwavelength features of the object. Due to the decaying nature in conventional materials, evanescent waves are permanently lost in the image region, resulting in the limited resolution of the imaging system. This is the well known diffraction limit, which defines the spatial resolution not smaller than half of the operating wavelength. To overcome the diffraction limit, it is necessary for the imaging system to interact with evanescent waves as the result of their amplitudes being enhanced or preserved.

For EM waves, Pendry proposed the idea of using metamaterials with negative refractive index to overcome the diffraction limit by exciting surface waves. Since then, much efforts have been made to achieve the super-resolution imaging based on the metamaterial concept. Ambati et al. found that negative mass of metamaterials is the necessary condition to support surface resonant states for acoustic waves. Consider two semi-infinite fluid material of mass density $\rho$, $\rho_0$ and modulus $\kappa$, $\kappa_0$. The surface resonant states represent the perturbations with pressures maximum at the interface and exponentially decaying in the direction perpendicular to the interface. By use of continuous conditions of pressure and normal velocity fields at the two interfaces, the conditions for surface states can be derived as

$$\frac{k_x}{\rho} + \frac{k_{0x}}{\rho_0} = 0, \quad (28)$$

where $k_x^2 + k_y^2 = \omega^2 \rho / \kappa$ and $k_{0x}^2 + k_{0y}^2 = \omega^2 \rho_0 / \kappa_0$. In Eq. (28), it is found that negative mass $\rho < 0$ is the necessary condition of the surface resonant states. The evanescent wave enhancement by negative-mass metamaterials has been validated by numerical simulation and experiment.

Acoustic metamaterials with anisotropic mass provide an alternative approach to overcome the diffraction limit. Suppose a flat slab with a mass tensor $\tilde{\rho} = \text{diag} [\rho_\perp, \rho_\parallel]$, the dispersion relation is written as

$$\frac{k_x^2}{\rho_\perp} + \frac{k_y^2}{\rho_\parallel} = \frac{\omega^2}{\kappa}, \quad (29)$$
where $k_x$ and $k_y$ are respectively the wave vectors perpendicular and parallel to the slab surface. When $\rho_1 \to \infty$, and $\rho_\perp$ is around the mass density of the background material, evanescent waves could be converted to propagating waves with the same wave vector $k = \omega\sqrt{\rho_\perp/\kappa}$. In this case, all evanescent wave components will be transferred to the output side of the slab lens to form super-resolution images. In practice, the infinite mass can be realized in steel or brass slabs by the extremely large impedance mismatch between the metal slab and surrounding air.\textsuperscript{50,51} For complete transmission of evanescent waves through the slabs, the Fabry-Pérot resonant condition $k_0h = n\pi (n = 1, 2, \ldots)$ should be satisfied.\textsuperscript{51,52} Experiments based on such mechanism have been performed\textsuperscript{51} and the spatial resolution $\lambda/50$ of deep subwavelength is achieved.

Zhou and Hu found that evanescent waves can be efficiently transferred based on zero effective mass.\textsuperscript{53} The operating frequency of the lens will depend on the way how zero effective mass is realized. Based on the metamaterial technology, a lens with zero effective mass has been designed. The designed lens consists of solid slabs with a periodic array of slits partially filled by elastic layers, as shown in Fig. 15. The solid slabs are all fixed, so that infinite effective mass of the lens in the vertical direction is strictly satisfied. In the horizontal direction, the clamped elastic layers inside the slits undergo resonances at their lowest eigenfrequency. The resonant vibration of clamped layers will result in effective mass following the Drude-form expression $\rho_\perp = \rho_0 (1 - \omega_0^2/\omega^2)$.\textsuperscript{22} Zero effective mass can be realized at the cutoff frequency $\omega_0$. To verify the imaging effect, two monopole line sources separated by around $\lambda_0/9$ are placed in front of the designed lens and the image plane is taken closely behind the lens. Figure 16 shows the contour plots of pressure amplitude distributions in the image plane around the designed frequency 1 898 Hz. It can be seen that the distance between two sources, which is less than the half wavelength, can be clearly distinguished from the images. These results demonstrate that the metamaterial superlens can create the images with spatial resolution beyond the diffraction limit.

IV. CONCLUSION

Elastic metamaterials with local resonances could result in negative values of effective dynamic mass, bulk or shear modulus. The underlying mechanisms have been discovered by both mass-spring and continuum material models. Cloaking and superlensing effects of elastic metamaterials are introduced as two examples of their interesting applications, and the principle and design methods are explained in details. It is expected that more applications of elastic metamaterials will be anticipated in the near future.

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