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Eshelby tensors for an ellipsoidal inclusion in a microstretch material

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Abstract

Eshelby tensors for an ellipsoidal inclusion in a microstretch material are derived in analytical form, involving only one-dimensional integral. As micropolar Eshelby tensor, the microstretch Eshelby tensors are not uniform inside of the ellipsoidal inclusion. However, different from micropolar Eshelby tensor, it is found that when the size of inclusion is large compared to the characteristic length of microstretch material, the microstretch Eshelby tensor cannot be reduced to the corresponding classical one. The reason for this is analyzed in details. It is found that under a pure hydrostatic loading, the bulk modulus of a microstretch material is not the same as the one in the corresponding classical material. A modified bulk modulus for the microstretch material is proposed, the microstretch Eshelby tensor is shown to be reduced to the modified classical Eshelby tensor at large size limit of inclusion. The fully analytical expressions of microstretch Eshelby tensors for a cylindrical inclusion are also derived.

Keywords: Eshelby tensor; Microstretch; Ellipsoidal inclusion; Circular cylindrical inclusion

1. Introduction

Microcontinuum theory summarized in the monograph by Eringen (1999) is believed to be a potential candidate to bridge the gap between a system of discrete atoms and a continuum. This theory incorporates independent deformations of the microstructure inside of a material point, while the theory itself remains in a continuum formulation. There are a number of microcontinuum theories, namely couple stress, micropolar, microstretch and micromorphic (Eringen, 1999). These theories impose more or less constraints on the motion of microstructure inside of a material point. Their connection with atomic information and their applicability are recently discussed by Chen et al. (2004). In microstretch theory which we will discuss in this paper, it is

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assumed that the microstructure of each material point can undergo independently expansion or contraction in addition to translation and rigid rotation. This theory is a generalization of micropolar theory, in which the microstructure can have only translation and rigid rotation. It is believed that microstretch theory is more suitable for materials with deformable microstructures, such as materials with pores (Eringen, 1999; Liu and Hu, 2004).

To predict overall property for composite materials, the inclusion theory (Mura, 1982) was extensively utilized based on the pioneering work of Eshelby (1957) for an ellipsoidal inclusion in a classical material. As discussed by Hu et al. (2005), when the size of reinforced particle is comparable to intrinsic length of matrix material, the nonlocal effect must be considered in a proper theoretical formulation. Homogenization techniques for a micropolar composite were recently established (Sharma and Dasgupta, 2002; Xun et al., 2004; Liu and Hu, 2005; Ma and Hu, 2006). They are all based on the micropolar Eshelby tensors for spherical or cylindrical inclusions given by Cheng and He (1995, 1997) and for a general ellipsoidal inclusion derived by Ma and Hu (2006). Size-dependence of the overall elastic and plastic properties has been well predicted and it is compared favorably with experiment (Liu and Hu, 2004).

However, since only translation and rigid rotation of microstructure are taken into account in micropolar theory, the size-dependence of bulk modulus and of plasticity under hydrostatic loading cannot be predicted within the homogenization theory for micropolar composites. To remedy this, homogenization theory for microstretch composites is believed to be suitable way. The building block of homogenization theory for microstretch composites, *inclusion problem*, is only recently studied for a spherical inclusion (Liu and Hu, 2004; Kiris and Inan, 2005). The microstretch Eshelby tensor for a general ellipsoidal inclusion is not available in literature. So the objective of this manuscript is to derive microstretch Eshelby tensors for a general ellipsoidal inclusion. As a special case, the microstretch Eshelby tensors for a circular cylindrical inclusion will also be derived in a fully analytical form. The manuscript will be arranged as follows: In Section 2, a brief theory for a microstretch material will be recalled. In Section 3, analytical expressions of microstretch Eshelby tensors for a general ellipsoidal inclusion will be derived. The average microstretch Eshelby tensors over the ellipsoidal domain and the circular cylindrical domain will be given in Section 4. The characteristics of these Eshelby tensors will be examined through numerical examples, and this will be presented in Section 5. The property of these Eshelby tensors when the size of inclusion tends to infinity will be discussed in Section 6, and followed by some conclusions.

Index notation for a tensor (or vector) is adopted throughout this paper, except some vector representations appear in bold letter as used for convenience.

2. Basic equations for centrosymmetric and isotropic microstretch material

For a centrosymmetric and isotropic microstretch continuum (invariant with respect to coordinate rotations and inversions (Lakes and Benedict, 1982)), the governing equations are given by Eringen (1999)

geometrical relations:

$$\varepsilon_{ji} = u_{i,j} + e_{ijk}\phi_k, \quad \kappa_{ji} = \phi_{i,j}, \quad \zeta_i = \theta_{,i} \tag{1a}$$

balance equations:

$$\sigma_{ji,j} + f_i = 0, \quad m_{ji,j} + e_{ikl}\sigma_{kl} + l_i = 0, \quad p_{k,k} - s + l = 0 \tag{1b}$$

constitutive equations:

$$\sigma_{ji} = C_{jikl}\varepsilon_{kl} + \delta_{ji}\lambda_0\theta, \quad m_{ji} = D_{jikl}\kappa_{kl}, \quad p_i = \eta\theta_{,i} \quad s = \lambda_0\varepsilon_{kk} + b\theta \tag{1c}$$

where u_i and ϕ_i are the displacement and microrotation, respectively, and θ represents the corresponding microstretch. ε_{ji} and κ_{ji} are respectively the strain and torsion tensors introduced in micropolar theory, ζ_i is the space gradient of the microstretch. e_{ijk} is the third order permutation tensor. σ_{ji} and m_{ji} are the asymmetric stress and couple stress tensors, p_k and s are the new stress quantities introduced in microstretch theory. Thermodynamically, they are conjugate to ζ_i and θ. f_i, l_i and l are the generalized body forces.

C_{jikl} and D_{jikl} are the elasticity tensors of the isotropic microstretch material, and they have the following form (Eringen, 1999):

$$C_{jikl} = \lambda\delta_{ij}\delta_{kl} + (\mu+\kappa)\delta_{jk}\delta_{il} + (\mu-\kappa)\delta_{ik}\delta_{jl} \tag{2a}$$

$$D_{jikl} = \alpha\delta_{ij}\delta_{kl} + (\beta+\gamma)\delta_{jk}\delta_{il} + (\beta-\gamma)\delta_{ik}\delta_{jl} \tag{2b}$$

where μ, λ are the classical Lamé constants and κ, γ, β, α are the elastic constants related to micropolar property, λ_0, η and b are the new elastic constants due to microstretch theory. The range of their values has been discussed by Eringen (1999).

Due to the dimensional difference between the two sets of moduli, four intrinsic characteristic lengths can be defined for an isotropic elastic microstretch material, they can be defined as

$$l_1 = (\beta/\mu)^{1/2}, \quad l_2 = (\gamma/\mu)^{1/2}, \quad l_3 = (\alpha/\mu)^{1/2}, \quad l_4 = (\eta/\mu)^{1/2} \tag{3}$$

Following Liu and Hu (2004), in an infinitely extended microstretch body, the following impulse body forces at the position $\mathbf{x}'$ are prescribed:

$$\mathbf{f} = \mathbf{F}\delta(\mathbf{x}-\mathbf{x}'), \quad \mathbf{l} = \mathbf{L}\delta(\mathbf{x}-\mathbf{x}'), \quad l = L\delta(\mathbf{x}-\mathbf{x}') \tag{4}$$

The fundamental solutions of microstretch theory are (Liu and Hu, 2004)

$$u_i(\mathbf{x}) = [G^1_{ij}(\mathbf{x}-\mathbf{x}') + G^s_{ij}(\mathbf{x}-\mathbf{x}')]F_j + G^2_{ij}(\mathbf{x}-\mathbf{x}')L_j + G_i(\mathbf{x}-\mathbf{x}')L \tag{5a}$$

$$\phi_i(\mathbf{x}) = H^1_{ij}(\mathbf{x}-\mathbf{x}')F_j + H^2_{ij}(\mathbf{x}-\mathbf{x}')L_j \tag{5b}$$

$$\theta(\mathbf{x}) = \Theta_j(\mathbf{x}-\mathbf{x}')F_j + \Theta(\mathbf{x}-\mathbf{x}')L \tag{5c}$$

where G^1_{ij}, $G^2_{ij} = H^1_{ij}$ and H^2_{ij} are the Green's functions for micropolar theory, which have been provided by Sandru (1966). G^s_{ij}, G_i, Θ_i and Θ are the additional Green's functions related to microstretch property (Liu and Hu, 2004). Their analytical expressions are listed in Appendix A.

3. Inclusion problem

Considering an ellipsoidal inclusion Ω in an infinite centrosymmetric and isotropic microstretch material, a uniform asymmetric eigenstrain ε^*_{ji}, an eigentorsion κ^*_{ji} and an eigenmicrostretch-gradient $\zeta^*_k(\mathbf{x})$ are prescribed in the inclusion. Here the inclusion means that its material constants are the same as the surrounding matrix, as introduced by Mura (1982). It can be shown that the consequence of these eigendeformations can be simulated by distributed body loads. With the help of Green's functions for a microstretch material, the induced displacement, rotation and microstretch due to the prescribed eigendeformations can be expressed as

$$\begin{aligned} u_n(\mathbf{x}) = \int_V \Big\{ & -C_{jikl}\varepsilon^*_{kl}(\mathbf{x}')(G^1_{in,j}(\mathbf{x}-\mathbf{x}') + G^s_{in,j}(\mathbf{x}-\mathbf{x}')) + 2\kappa e_{ikl}\varepsilon^*_{kl}(\mathbf{x}')H^1_{in}(\mathbf{x}-\mathbf{x}') \\ & + D_{jikl}\kappa^*_{kl}(\mathbf{x}')H^1_{in,j}(\mathbf{x}-\mathbf{x}') - \lambda_0\varepsilon^*_{rr}(\mathbf{x}')\Theta_n(\mathbf{x}-\mathbf{x}') + \eta\zeta^*_r(\mathbf{x}')\Theta_{n,r}(\mathbf{x}-\mathbf{x}')\Big\}\,\mathrm{d}\mathbf{x}' \end{aligned} \tag{6a}$$

$$\phi_n(\mathbf{x}) = \int_V \Big\{ C_{jikl}\varepsilon^*_{kl}(\mathbf{x}')G^2_{in,j}(\mathbf{x}-\mathbf{x}') - 2\kappa e_{ikl}\varepsilon^*_{kl}(\mathbf{x}')H^2_{in}(\mathbf{x}-\mathbf{x}') - D_{jikl}\kappa^*_{kl}(\mathbf{x}')H^2_{in,j}(\mathbf{x}-\mathbf{x}')\Big\}\,\mathrm{d}\mathbf{x}' \tag{6b}$$

$$\theta(\mathbf{x}) = \int_V \{ C_{jikl}\varepsilon^*_{kl}(\mathbf{x}')G_{i,j}(\mathbf{x}-\mathbf{x}') + \lambda_0\varepsilon^*_{rr}(\mathbf{x}')\Theta(\mathbf{x}-\mathbf{x}') - \eta\zeta^*_r(\mathbf{x}')\Theta_{,r}(\mathbf{x}-\mathbf{x}')\}\,\mathrm{d}\mathbf{x}' \tag{6c}$$

where $_{,j}$ means $\frac{\partial}{\partial x_j}$.

With the help of the expressions of Green's functions listed in Appendix A, Eq. (6c) can be written as

$$\theta(\mathbf{x}) = I^s_{kl}(\mathbf{x})\varepsilon^*_{kl} + T^s_k(\mathbf{x})\zeta^*_k \tag{7a}$$

It is noted that $\theta(\mathbf{x})$ is not only a displacement quantity, but also a strain measure for the microstretch material (see, Eq. (1c)).

According to the constitutive equation, differentiating the both sides of Eq. (6), the induced strain, torsion and microstretch-gradient can be written as (Liu and Hu, 2004)

$$\varepsilon_{mn}(\mathbf{x}) = K_{mnkl}(\mathbf{x})\varepsilon^*_{kl} + L_{mnkl}(\mathbf{x})\kappa^*_{kl} + N^s_{mnk}(\mathbf{x})\zeta^*_k \tag{7b}$$

$$\kappa_{mn}(\mathbf{x}) = \widehat{K}_{mnkl}(\mathbf{x})\varepsilon^*_{kl} + \widehat{L}_{mnkl}(\mathbf{x})\kappa^*_{kl} \tag{7c}$$

$$\zeta_n(\mathbf{x}) = K^s_{nkl}(\mathbf{x})\varepsilon^*_{kl} + N^s_{nk}(\mathbf{x})\zeta^*_k \tag{7d}$$

where

$$K_{mnji}(\mathbf{x}) = K^1_{mnji}(\mathbf{x}) + K^s_{mnji}(\mathbf{x})$$
$$K^1_{mnji}(\mathbf{x}) = I^c_{nji,m}(\mathbf{x}) + I_{nji,m}(\mathbf{x}) - e_{lmn}\widehat{I}_{lji}(\mathbf{x}), \quad K^s_{mnji}(\mathbf{x}) = I^s_{nji,m}(\mathbf{x})$$
$$L_{mnji}(\mathbf{x}) = J_{nji,m}(\mathbf{x}) - e_{lmn}\widehat{J}_{lji}(\mathbf{x}), \quad N^s_{mnk}(\mathbf{x}) = T^s_{nk,m}(\mathbf{x})$$
$$\widehat{K}_{mnji}(\mathbf{x}) = \widehat{I}_{nji,m}(\mathbf{x}), \quad \widehat{L}_{mnji}(\mathbf{x}) = \widehat{J}_{nji,m}(\mathbf{x})$$
$$K^s_{nkl}(\mathbf{x}) = I^s_{kl,n}(\mathbf{x}), \quad N^s_{nk}(\mathbf{x}) = T^s_{k,n}(\mathbf{x})$$

The tensors K_{mnkl}, L_{mnkl}, N^s_{mnk}, $\widehat{K}_{mnkl}$, $\widehat{L}_{mnkl}$, K^s_{nkl}, N^s_{nk} are called microstretch Eshelby tensors (Liu and Hu, 2004). K^1_{mnkl}, L_{mnkl}, $\widehat{K}_{mnkl}$, $\widehat{L}_{mnkl}$ are the corresponding micropolar Eshelby tensors which have been given by Cheng and He (1995); K^s_{mnkl}, N^s_{mnk}, K^s_{nkl}, N^s_{nk} are the additional Eshelby tensors due to the microstretch effect derived by Liu and Hu (2004). The detailed expressions of $I^c_{nji}(\mathbf{x})$, $I_{nji}(\mathbf{x})$, $\widehat{I}_{lji}(\mathbf{x})$, $I^s_{nji}(\mathbf{x})$, $J_{nji}(\mathbf{x})$, $\widehat{J}_{lji}(\mathbf{x})$, $I^s_{kl}(\mathbf{x})$, $T^s_{kl}(\mathbf{x})$, $T^s_k(\mathbf{x})$ are listed in Appendix B.

It can be seen from Appendix B that evaluation of microstretch Eshelby tensors depends on the following three potential functions and their derivatives, which are defined by

$$\psi(\mathbf{x}) = \frac{1}{4\pi}\int_\Omega r\,\mathrm{d}\mathbf{x}', \quad \phi(\mathbf{x}) = \frac{1}{4\pi}\int_\Omega \frac{1}{r}\,\mathrm{d}\mathbf{x}', \quad M(\mathbf{x},k) = \frac{1}{4\pi}\int_\Omega \frac{\mathrm{e}^{-r/k}}{r}\,\mathrm{d}\mathbf{x}' \tag{8}$$

where $r = |\mathbf{x} - \boldsymbol{x}'|$.

The first and second integrals appeared in Eq. (8) are the same as in classical Eshelby tensor (Mura, 1982), and they have been evaluated analytically by Eshelby (1957) for a general ellipsoidal inclusion. Then, computation of the third one is the key point for evaluating the microstretch Eshelby tensors.

For a spherical inclusion, the last integral of Eq. (8) has been provided analytically by Cheng and He (1995). So the analytical expressions of microstretch Eshelby tensors for a spherical inclusion can be obtained (Liu and Hu, 2004). However for a general ellipsoidal inclusion, it cannot be evaluated in a fully analytical form. Ma and Hu (2006) have reduced it to the following form, which involves only one-dimensional integral:

$$M(\mathbf{x},k) = \frac{1}{4\pi}\int_\Omega \frac{\mathrm{e}^{-r/k}}{r}\,\mathrm{d}\mathbf{x}' = k^2 - k^2\frac{a_3}{2}\int_0^\infty (D\cdot A)\,\mathrm{d}u \tag{9}$$

where the parameters in Eq. (9) are defined as

$$D = \frac{1}{(u+a_3^2)^{3/2}}\left(1 + \frac{a_1}{k}\sqrt{\frac{u+a_3^2}{u+a_1^2}}\right)\exp\left(-\frac{a_1}{k}\sqrt{\frac{u+a_3^2}{u+a_1^2}}\right)$$
$$A = I_0(B\rho)\cosh(Cx_3), \quad B = \frac{1}{k}\sqrt{\frac{u}{u+a_1^2}}, \quad C = \frac{a_1}{k\sqrt{u+a_1^2}}$$
$$u = a_3^2\tan^2\theta, \quad \rho = \sqrt{x_1^2 + x_2^2}$$

In the above expression, I_M is the Mth order modified Bessel function of the first kind, a_1 is the half short axis of the ellipsoid and a_3 is its half major axis. The major axis of the ellipsoid lines along the axis x_3.

The derivatives of Eq. (9) are listed in Appendix C. So with the help of Eq. (9), all the microstretch Eshelby tensors for an ellipsoidal inclusion can be obtained.

4. Average microstretch Eshelby tensors

As in the case for Eshelby tensor in a micropolar material, the microstretch Eshelby tensors are not uniform inside of an ellipsoidal inclusion. So in the following, the microstretch Eshelby tensors are averaged over the ellipsoidal inclusion. The average Eshelby tensors are useful in predicting the overall properties of composite materials. In the following, for simplicity, we examine only the following two tensors: one is the symmetric part $K_{ijkl}^{\rm sym}$ of the Eshelby tensor K_{ijkl}, defined by

$$K_{ijmn}^{\rm sym} = \frac{1}{4}\left(K_{ijmn} + K_{ijnm} + K_{jimn} + K_{jinm}\right) \tag{10}$$

The other is I_{kl}^s, which relates the eigenstrain ε_{ij}^* to the induced microstretch $\theta(\mathbf{x})$. From Appendices B and C, it is seen that tensor I_{kl}^s is a symmetric quantity. The other microstretch Eshelby tensors can be evaluated in the same way.

(a)Average microstretch Eshelby tensors for an ellipsoidal inclusion

With the help of the result in Appendix B and Eqs. (5), (6), we find that the expressions for L_{mnkl}, $\widehat{K}_{mnkl}$, N_{mnk}^s, K_{nkl}^s and T_k^s have only odd order terms of the argument $\mathbf{x}$. So their integration over a symmetric domain vanishes. Therefore for a general ellipsoidal inclusion, the following properties hold:

$$\langle L_{mnkl}\rangle_\Omega = \langle \widehat{K}_{mnkl}\rangle_\Omega = \langle N_{mnk}^s\rangle_\Omega = \langle K_{nkl}^s\rangle_\Omega = \langle T_k^s\rangle_\Omega = 0 \tag{11}$$

where $\langle \bullet \rangle_\Omega$ means the volume average of the said quantity over the inclusion domain.

Eq. (11) has been proven for a spherical inclusion by Liu and Hu (2004). The result of Eq. (11) means that the average microstretch Eshelby relations (Eq. (7)) are uncoupled. That is,

$$\begin{aligned} &\langle \varepsilon_{mn}(\mathbf{x})\rangle_\Omega = \langle K_{mnkl}(\mathbf{x})\rangle_\Omega \varepsilon_{kl}^*, \quad \langle \kappa_{mn}(\mathbf{x})\rangle_\Omega = \langle \widehat{L}_{mnkl}(\mathbf{x})\rangle_\Omega \kappa_{kl}^* \\ &\langle \zeta_n(\mathbf{x})\rangle_\Omega = \langle N_{nk}^s(\mathbf{x})\rangle_\Omega \zeta_k^*, \quad \langle \theta(\mathbf{x})\rangle_\Omega = \langle I_{kl}^s(\mathbf{x})\rangle_\Omega \varepsilon_{kl}^* \end{aligned} \tag{12}$$

The average symmetric part $\langle K_{mnkl}^{\rm sym}\rangle_\Omega$ of the Eshelby tensor $\langle K_{mnkl}\rangle_\Omega$ relates the symmetric part of strain to the symmetric part of the eigenstrain by

$$\langle \varepsilon_{mn}^{\rm sym}\rangle_\Omega = \langle K_{mnkl}^{\rm sym}\rangle_\Omega \varepsilon_{kl}^{*\rm sym} \tag{13}$$

Following the method proposed by Ma and Hu (2006), and with help of Eq. (7), the average microstretch Eshelby tensors for a general ellipsoidal inclusion can then be computed. It will be performed in Section 5.

(b)Average microstretch Eshelby tensors for a circular cylindrical inclusion

With the analytical expression for $M(\mathbf{x}, k)$ given by Cheng and He (1997) for a cylindrical inclusion and the general Eqs. (6), (7), the average microstretch Eshelby tensors for a circular cylindrical inclusion can be evaluated in a fully analytical from. After some mathematical manipulation, the average microstretch Eshelby tensors for a cylindrical inclusion are given by

$$\langle K_{\alpha\beta\gamma\rho}\rangle_\Omega = T_1\delta_{\alpha\beta}\delta_{\gamma\rho} + (T_2 + T_3)\delta_{\alpha\gamma}\delta_{\beta\rho} + (T_2 - T_3)\delta_{\alpha\rho}\delta_{\beta\gamma} \tag{14a}$$

$$\langle \widehat{L}_{\alpha 3\gamma 3}\rangle_\Omega = Q_{33}\delta_{\alpha\gamma} \tag{14b}$$

$$\langle N_{\beta\gamma}^s\rangle_\Omega = N^s\delta_{\beta\gamma} \tag{14c}$$

$$\langle I_{\gamma\rho}^s\rangle_\Omega = I^s\delta_{\gamma\rho} \tag{14d}$$

where indices α, β, γ, ρ range from 1 to 2, and

$$T_1 = \frac{\lambda-\mu}{4(\lambda+2\mu)} + \frac{\kappa}{2(\kappa+\mu)} I_1(a/h)K_1(a/h) - \frac{3\mu\lambda_0^2}{4A^s(\lambda+2\mu)} + \frac{3\mu\eta\lambda_{0^2}}{2A^{s^2}p^2} I_1(a/p)K_1(a/p)$$

$$T_2 = \frac{\lambda+3\mu}{4(\lambda+2\mu)} - \frac{\kappa}{2(\kappa+\mu)} I_1(a/h)K_1(a/h) + \frac{\mu\lambda_0^2}{4A^s(\lambda+2\mu)} - \frac{\mu\eta\lambda_0^2}{2A^{s^2}p^2} I_1(a/p)K_1(a/p)$$

$$T_3 = \frac{1}{2} - \frac{\mu}{\kappa + \mu} I_1(a/h)K_1(a/h)$$
$$Q_{33} = I_1(a/h)K_1(a/h)$$
$$N^s = I_1(a/p)K_1(a/p)$$
$$I^s = \frac{\mu\lambda_0}{A^s} - \frac{2\mu\lambda_0}{A^s} I_1(a/p)K_1(a/p)$$

In the above expression, a denotes the radius of the cylinder, I_1, K_1 are the first order modified Bessel function of the type I and II, respectively. Other constants appear in the above expression are

$$h = \sqrt{\frac{(\mu+\kappa)(\gamma+\beta)}{4\mu\kappa}}, \quad p = \sqrt{\frac{\eta(\lambda+2\mu)}{(\lambda+2\mu)b-\lambda_0^2}}, \quad A^s = (\lambda+2\mu)b - \lambda_0^2 \tag{15}$$

The average symmetric part $\langle K^{\mathrm{sym}}_{\alpha\beta\gamma\rho}\rangle_\Omega$ of the Eshelby tensor $\langle K_{\alpha\beta\gamma\upsilon}\rangle_\Omega$ is given by

$$\langle K^{\mathrm{sym}}_{\alpha\beta\gamma\rho}\rangle_\Omega = T_1\delta_{\alpha\beta}\delta_{\gamma\rho} + T_2(\delta_{\alpha\gamma}\delta_{\beta\rho} + \delta_{\alpha\rho}\delta_{\beta\gamma}) \tag{16}$$

For completeness, the analytical expressions of the average microstretch Eshelby tensors for a spherical inclusion are also listed for comparison (Liu and Hu, 2004)

$$\langle K^{\mathrm{sym}}_{mnkl}\rangle_\Omega = T_1\delta_{mn}\delta_{kl} + T_2(\delta_{mk}\delta_{nl} + \delta_{ml}\delta_{nk}), \quad \langle N^s_{kl}\rangle_\Omega = N^s\delta_{kl}, \quad \langle I^s_{kl}\rangle_\Omega = I^s\delta_{kl} \tag{17}$$

Indices i, j, k, l range from 1 to 3, where

$$T_1 = \frac{3\lambda-2\mu}{15(\lambda+2\mu)} + \frac{2h(a+h)\kappa}{5a^3(\kappa+\mu)}\Gamma(h) - \frac{8\mu\lambda_{0^2}}{15A^s(\lambda+2\mu)} + \frac{8(a+p)\mu\eta\lambda_0^2}{5a^3A^{s^2}p}\Gamma(p)$$
$$T_2 = \frac{3\lambda+8\mu}{15(\lambda+2\mu)} - \frac{3h(a+h)\kappa}{5a^3(\kappa+\mu)}\Gamma(h) + \frac{2\mu\lambda_0^2}{15A^s(\lambda+2\mu)} - \frac{2(a+p)\mu\eta\lambda_0^2}{5a^3A^{s^2}p}\Gamma(p)$$
$$N^s = \frac{p(a+p)}{a^3}\Gamma(p)$$
$$I^s = \frac{4\mu\lambda_0}{3A^s} - \frac{4\mu\lambda_0 p(a+p)}{a^3A^s}\Gamma(p)$$
$$\Gamma(y) \equiv \mathrm{e}^{-a/y}[a\cosh(a/y) - y\sinh(a/y)]$$

In the expressions of T_1, T_2 for spherical and cylindrical inclusions, the first term is the classical part, the second item is due to the micropolar effect, and the last two terms come from the coupling microstretch effect. It is clear that when the size of inclusion tends to infinity, contrary to micropolar Eshelby tensors, the size-dependent terms tend to zero, T_1, T_2 cannot be reduced to the classical ones. The reason will be explored in Section 6. In the section followed, some numerical examples will be given to illustrate the property of the derived microstretch Eshelby tensors.

5. Numerical examples

In this section, we assume $l_1 = l_2 = l_3 = l_4 = l_\mathrm{m}$, the other material constants used in the computation are $\lambda = 50$ GPa, $\mu = 26$ GPa, $\kappa = 13$ GPa, $\lambda_0 = 25$ GPa, $b = 26$ GPa, $l_\mathrm{m} = 10$ μm.

The variations of $\langle K^{\mathrm{sym}}_{1111}\rangle_\Omega$ and $\langle K^{\mathrm{sym}}_{1122}\rangle_\Omega$ as function of the aspect ratio of inclusion are shown in Figs. 1 and 2, respectively. For comparison, the corresponding classical Eshelby tensor and the micropolar Eshelby tensor are also included. The exact values of the average microstretch Eshelby tensors for a spherical inclusion and a cylindrical inclusion (Eqs. (16) and (17)) are also included in the figures in order to access the accuracy of the numerical computation. The size of inclusion is set to be $a_1 = l_\mathrm{m}$. We found that our numerical results agree very well with the exact results in the case of the spherical and cylindrical inclusions.

The variations of the components $\langle I^s_{11}\rangle_\Omega$ and $\langle I^s_{33}\rangle_\Omega$ of the tensor $\langle I^s_{kl}\rangle_\Omega$ as function of the aspect ratio of inclusion are shown in Fig. 3. The exact values of the microstretch average Eshelby tensor for a spherical inclusion or a cylindrical inclusion are also included. It is found that for a spherical inclusion, the numerical

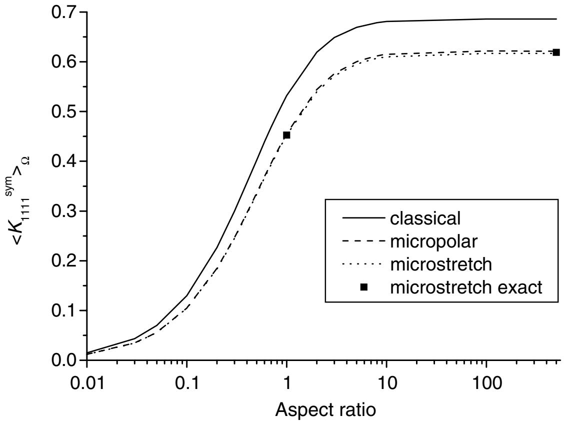

Fig. 1. Variation of $\langle K^{\mathrm{sym}}_{1111}\rangle_{\Omega}$ as function of aspect ratio of inclusion.

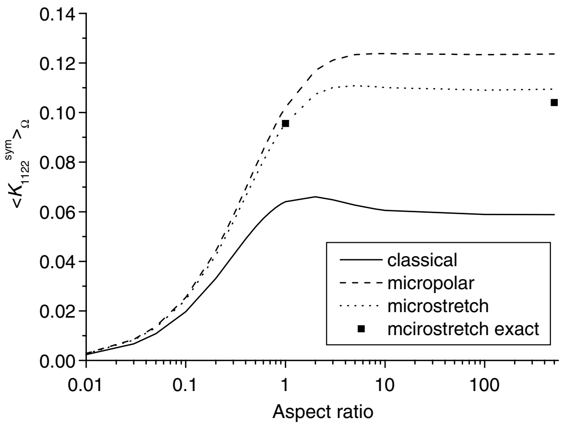

Fig. 2. Variation of $\langle K^{\mathrm{sym}}_{1122}\rangle_{\Omega}$ as function of aspect ratio of inclusion.

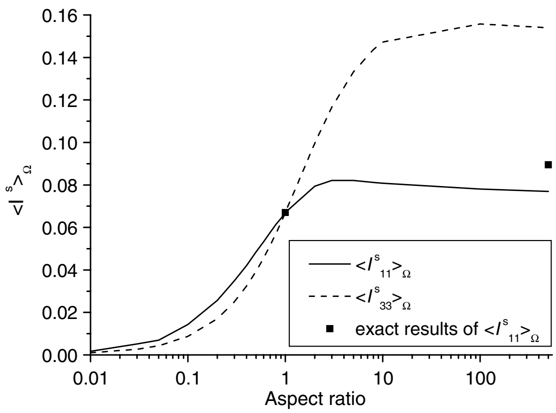

Fig. 3. Variations of $\langle I^{s}_{11}\rangle_{\Omega}$ and $\langle I^{s}_{33}\rangle_{\Omega}$ as function of aspect ratio of inclusion.

results have a very good agreement with the exact result. However for a circular cylindrical inclusion, due to the finite size of the inclusion used in the numerical computation, there is a little difference between the numerical and the exact results.

6. Discussion on average microstretch Eshelby tensors

As discussed in Section 4, unlike micropolar Eshelby tensors, the average microstretch Eshelby tensor cannot be reduced to the corresponding classical one when the size of inclusion tends to infinity. In the following discussion, we will examine the reason why the microstretch Eshelby tensor cannot be reduced to the classical one at large size limit of inclusion.

We will examine the response of a pure microstretch material under a hydrostatic loading. Following micropolar theory (Hu et al., 2005), the constitutive relations (Eq. (1c)) of a microstretch material can be rewritten as

$$\sigma'_{(ij)} = 2\mu\varepsilon'_{(ij)}, \quad \sigma_{\langle ij\rangle} = 2\kappa\varepsilon_{\langle ij\rangle}, \quad \bar{\sigma} = 3K\bar{\varepsilon} + \lambda_0\theta \tag{18a}$$

$$m'_{(ij)} = 2\beta\kappa\prime_{(ij)}, \quad m_{\langle ij\rangle} = 2\gamma\kappa_{\langle ij\rangle}, \quad \bar{m} = 3N\bar{\kappa} \tag{18b}$$

$$p_i = \eta\theta_{,i}, \quad s = 3\lambda_0\bar{\varepsilon} + b\theta \tag{18c}$$

where $\sigma'_{(ij)}$, $\sigma_{\langle ij\rangle}$, $\bar{\sigma}$ $(\equiv \sigma_{ii}/3)$ and $\varepsilon'_{(ij)}$, $\varepsilon_{\langle ij\rangle}$, $\bar{\varepsilon}$ $(\equiv \varepsilon_{ii}/3)$ denote separately the deviatoric symmetric, anti-symmetric and hydrostatic parts of stress and strain tensors, and similar notations for the couple-stress and torsion tensors. $K = \lambda + 2\mu/3$ is the bulk modulus. and $N = \alpha + 2\beta/3$, which can be interpreted as the corresponding stiffness measure for torsion.

If such a microstretch material is under a pure hydrostatic loading $\bar{\sigma}$, we find that

$$\bar{\sigma} = 3\left(\lambda - \frac{\lambda_0^2}{b} + \frac{2}{3}\mu\right)\bar{\varepsilon} \tag{19a}$$

for a spherical material sample, and

$$\bar{\sigma} = 2\left(\lambda - \frac{\lambda_0^2}{b} + \mu\right)\bar{\varepsilon} \tag{19b}$$

for a cylindrical material sample (with $\bar{\sigma} \equiv \sigma_{\alpha\alpha}/2, \ \bar{\varepsilon} \equiv \varepsilon_{\alpha\alpha}/2$).

It can be seen from Eq. (19) that if $\lambda_0 = 0$, the micropolar field is uncoupled with microstretch, the micropolar results can be found, which are identical to those for the classical material. However, for the microstretch material ($\lambda_0 \neq 0$), the bulk modulus is not the same as the classical material. That is reason why at large size limit of inclusion, the microstretch Eshelby tensor cannot be reduced to the classical Eshelby tensor. From Eq. (19), we can define a modified lame constant λ^{m} by

$$\lambda^m = \lambda - \frac{\lambda_0^2}{b} \tag{20}$$

With this modified lame constant, it can be expected that when the size of inclusion tends to infinity, the average microstretch Eshelby tensors $\langle K^{\mathrm{sym}}_{mnkl}\rangle_\Omega$ will be reduced to the classical Eshelby tensor with the material constants $(\mu, \lambda^{\mathrm{m}})$ instead of (μ, λ). If we use Eq. (20) to substitute the λ in the expressions of the classical terms in the average microstretch Eshelby tensors (Eq. (17)), after some mathematical manipulations, the following expressions are obtained:

$$T_1^m = \frac{3\lambda - 2\mu}{15(\lambda + 2\mu)} - \frac{8\mu\lambda_0^2}{15A^s(\lambda + 2\mu)}, \quad T_2^m = \frac{3\lambda + 8\mu}{15(\lambda + 2\mu)} + \frac{2\mu\lambda_0^2}{15A^s(\lambda + 2\mu)} \tag{21}$$

Eq. (21) gives just the size-independent terms in the average microstretch Eshelby tensor when the size of inclusion tends to infinity, as expected from the above reasoning. The other two terms remain unchanged, which characterize the size-dependence of the microstretch Eshelby tensor.

The variations of $\langle K^{\mathrm{sym}}_{1111}\rangle_\Omega$ and $\langle K^{\mathrm{sym}}_{1122}\rangle_\Omega$ as function of the size of inclusion are shown in Figs. 4 and 5, respectively. The micropolar results are also included. It is found that when the size of inclusion approaches to the characteristic length of matrix material (l_{m}), the influence of the size of inclusion is more pronounced for both micropolar and microstretch theory. When the size of inclusion is large enough, the micropolar results is

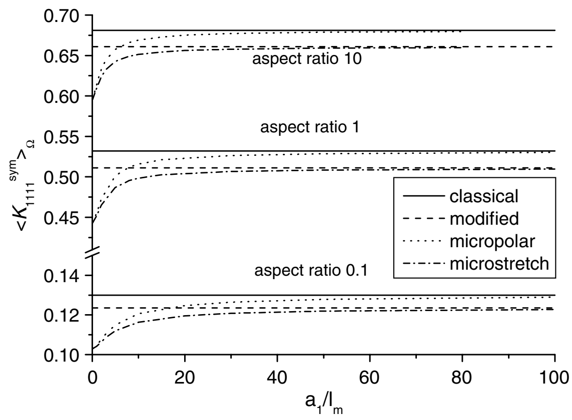

Fig. 4. Variation of $\langle K^{\mathrm{sym}}_{1111}\rangle_\Omega$ as function of size of inclusion for different aspect ratios 10, 1 and 0.1.

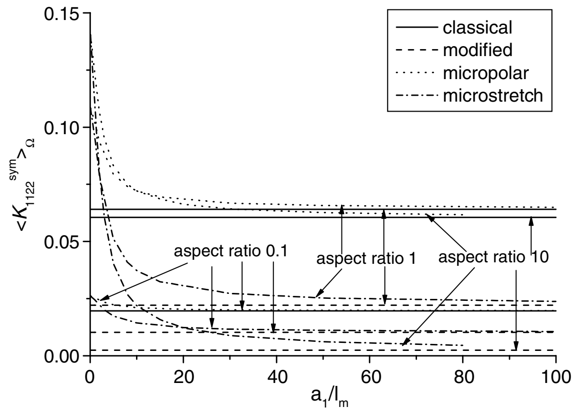

Fig. 5. Variation of $\langle K^{\mathrm{sym}}_{1122}\rangle_\Omega$ as function of size of inclusion for different aspect ratios 10, 1 and 0.1.

reduced to the classical one, however the microstretch Eshelby tensors tend to the modified one, as discussed previously.

The variations of $\langle I^s_{11}\rangle_\Omega$ and $\langle I^s_{33}\rangle_\Omega$ as function of the size of inclusion are also illustrated in Figs. 6 and 7, respectively. It is found that when the size of inclusion approaches to the characteristic length of matrix mate-

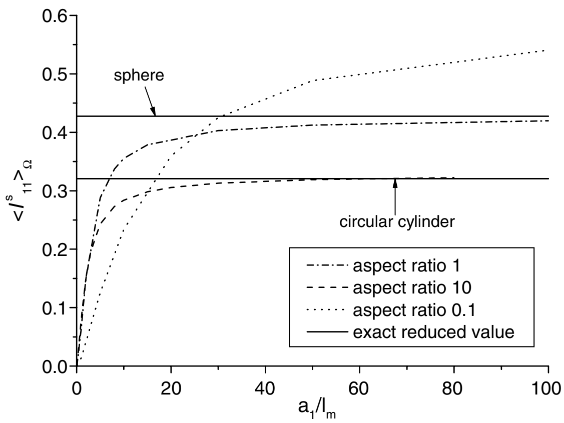

Fig. 6. Variations of $\langle I^s_{11}\rangle_\Omega$ as function of size of inclusion for different aspect ratios 10, 1 and 0.1.

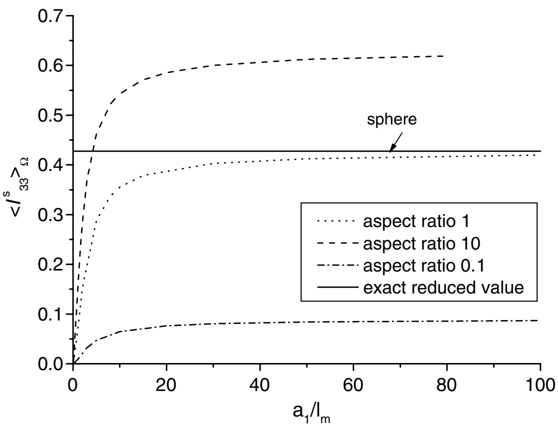

Fig. 7. Variations of $\langle I^s_{33}\rangle_\Omega$ as function of size of inclusion for different aspect ratios 10, 1 and 0.1.

rial (l_m), the influence of the size of inclusion is more significant. When the size of inclusion is large enough, both $\langle I^s_{11}\rangle_\Omega$ and $\langle I^s_{33}\rangle_\Omega$ approach to their asymptotic values. For a spherical inclusion, our numerical results tend to $4\mu\lambda_0/(3A^s)$, this is the exact value for average microstretch Eshelby tensor $\langle I^s_{kl}\rangle_\Omega$ for a spherical inclusion when the size of inclusion tends to infinity.

In order to avoid the "limit problem" at large size limit of inclusion, we can also consider a special case of microstretch theory by setting $\lambda_0 = 0$ and considering only η and b as the proper material constants. In this case, the displacement and rotation fields are uncoupled with the microstretch, the microstretch Eshelby tensors $\langle K\rangle_\Omega$ and $\langle \widehat{L}\rangle_\Omega$ are identical to those of the corresponding micropolar material, which are reduced to the classical results naturally. The tensors $\langle N^s\rangle_\Omega$ and $\langle I^s\rangle_\Omega$ due to the microstretch effect are checked to identically vanish when the size of inclusion tends to infinity.

7. Conclusions

We have therefore derived the microstretch Eshelby tensor for a general ellipsoidal inclusion. The expressions of these tensors involve only one-dimension integral, and can be easily computed. The fully analytical expressions of microstretch Eshelby tensors for a circular cylindrical are also obtained. The same as micropolar Eshelby tensors, the microstretch Eshelby tensors are not uniform inside of an ellipsoidal inclusion. It is shown that when the size of inclusion tends to infinity, the microstretch Eshelby tensor will be reduced to the modified classical Eshelby tensor. The obtained results will be useful for predicting the overall property of microstretch composites.

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Appendix A

The analytical expressions of Green's functions for an isotropic microstretch body are (Liu and Hu, 2004)

$$G^1_{ij}(\mathbf{x}-\mathbf{x}') = G^c_{ij}(\mathbf{x}-\mathbf{x}') - \frac{\kappa}{4\pi\mu(\mu+\kappa)}\left[h^2\left(\frac{1-\mathrm{e}^{-r/h}}{r}\right)_{,ij} + \delta_{ij}\frac{\mathrm{e}^{-r/h}}{r}\right]$$

$$G^2_{ij}(\mathbf{x}-\mathbf{x}') = H^1_{ij}(\mathbf{x}-\mathbf{x}') = -\frac{1}{8\pi\mu}e_{ijk}\left(\frac{1-\mathrm{e}^{-r/h}}{r}\right)_{,k}$$

$$H_{ij}^2(\mathbf{x}-\mathbf{x}') = \frac{1}{16\pi\mu}\left(\frac{1-\mathrm{e}^{-r/h}}{r}\right)_{,ij} + \frac{1}{16\pi\kappa}\left(\frac{\mathrm{e}^{-r/g}-\mathrm{e}^{-r/h}}{r}\right)_{,ij} + \frac{\mu+\kappa}{16\pi\mu\kappa h^2}\delta_{ij}\frac{\mathrm{e}^{-r/h}}{r}$$

$$G_{ij}^c(\mathbf{x}-\mathbf{x}') = \frac{1}{8\pi\mu}\left[2\delta_{ij}\frac{1}{r} - \frac{\lambda+\mu}{\lambda+2\mu}r_{,ij}\right] \text{ is the classical Green's function}$$

$$G_{ij}^s(\mathbf{x}-\mathbf{x}') = \frac{bp^2-\eta}{4\pi\eta(\lambda+2\mu)}\left\{\frac{1}{2}r_{,ij} + p^2\left(\frac{1-\mathrm{e}^{-r/p}}{r}\right)_{,ij}\right\}$$

$$G_i(\mathbf{x}-\mathbf{x}') = -\Theta_i(\mathbf{x}-\mathbf{x}') = \frac{\lambda_0 p^2}{4\pi\eta(\lambda+2\mu)}\left(\frac{1-\mathrm{e}^{-r/p}}{r}\right)_{,i}$$

$$\Theta(\mathbf{x}-\mathbf{x}') = \frac{1}{4\pi\eta}\frac{\mathrm{e}^{-r/p}}{r}$$

where $r = |\mathbf{x}-\mathbf{x}'|,\ g = \sqrt{\frac{(\alpha+2\beta)}{4\kappa}},\ h,\ p$ are given by Eq. (15).

Appendix B

$$I_{nji}^c = \frac{\lambda+\mu}{\lambda+2\mu}\psi_{,ijn}(\mathbf{x}) - \frac{\lambda}{\lambda+2\mu}\delta_{ij}\phi_{,n}(\mathbf{x}) - \delta_{in}\phi_{,j}(\mathbf{x}) - \delta_{jn}\phi_{,i}(\mathbf{x})$$

$$I_{nji} = \frac{2\kappa}{\mu+\kappa}\left[h^2\phi_{,ijn}(\mathbf{x}) - h^2M_{,ijn}(\mathbf{x},h) + \delta_{jn}M_{,i}(\mathbf{x},h)\right]$$

$$I_{nji}^s = \frac{2\mu\lambda_0^2}{A^s(\lambda+2\mu)}\left[-\frac{1}{2}\psi_{,nji}(\mathbf{x}) - p^2\phi_{,nji}(\mathbf{x}) + \delta_{ji}\phi_{,n}(\mathbf{x}) - \delta_{ji}M_{,n}(\mathbf{x},p) + p^2M_{,nji}(\mathbf{x},p)\right]$$

$$\begin{aligned}\widehat{I}_{nji}(\mathbf{x}) = &-\frac{1}{2\mu}[\lambda\delta_{ji}e_{nkl}\phi_{,kl}(\mathbf{x}) + \kappa e_{jik}\phi_{,kn}(\mathbf{x}) + (\mu+\kappa)e_{ink}\phi_{,kj}(\mathbf{x}) + (\mu-\kappa)e_{jnk}\phi_{,ki}(\mathbf{x})]\\ &+\frac{1}{2\mu}[\lambda\delta_{ji}e_{nkl}M_{,kl}(\mathbf{x},h) + (\mu+\kappa)e_{jik}M_{,kn}(\mathbf{x},h) + (\mu+\kappa)e_{ink}M_{,kj}(\mathbf{x},h) + (\mu-\kappa)e_{jnk}M_{,ki}(\mathbf{x},h)]\\ &-\frac{1}{2}e_{jik}M_{,kn}(\mathbf{x},g) - \frac{\mu+\kappa}{2\mu h^2}e_{nji}M(\mathbf{x},h)\end{aligned}$$

$$\begin{aligned}J_{nji}(\mathbf{x}) = &\frac{1}{2\mu}[(\beta+\gamma)e_{nik}\phi_{,jk}(\mathbf{x}) + (\beta-\gamma)e_{njk}\phi_{,ik}(\mathbf{x})] - \frac{1}{2\mu}[(\beta+\gamma)e_{nik}M_{,jk}(\mathbf{x},h) + (\beta-\gamma)e_{njk}M_{,ik}(\mathbf{x},h)]\\ &+\frac{\alpha}{2\mu}\delta_{ji}e_{nkl}[\phi_{,kl} - M_{,kl}(\mathbf{x},h)]\end{aligned}$$

$$\begin{aligned}\widehat{J}_{nji}(\mathbf{x}) = &-\frac{\beta}{2\mu}\phi_{,ijn}(\mathbf{x}) + \frac{\beta(\mu+\kappa)}{2\mu\kappa}M_{,ijn}(\mathbf{x},h) - \frac{1}{4\kappa}\left[\frac{\alpha}{g^2}\delta_{ji}M_{,n}(\mathbf{x},g) + 2\beta M_{,ijn}(\mathbf{x},g)\right]\\ &-\frac{\mu+\kappa}{4\mu\kappa h^2}[(\beta+\gamma)\delta_{in}M_{,j}(\mathbf{x},h) + (\beta-\gamma)\delta_{jn}M_{,i}(\mathbf{x},h)]\end{aligned}$$

$$I_{kl}^s(\mathbf{x}) = \frac{2\mu\lambda_0}{\eta(\lambda+2\mu)}\delta_{kl}M(\mathbf{x},p) + \frac{2\mu\lambda_0}{A^s}\left[\phi_{,kl}(\mathbf{x}) - M_{,kl}(\mathbf{x},p)\right]$$

$$T_{kl}^s(\mathbf{x}) = -\frac{\eta\lambda_0}{A^s}\left[\phi_{,kl}(\mathbf{x}) - M_{,kl}(\mathbf{x},p)\right]$$

$$T_k^s(\mathbf{x}) = -M_{,k}(\mathbf{x},p)$$

where A^s, h, p are given by Eq. (15), g is the same as in Appendix A.

Appendix C

The derivatives of Eq. (9) are given as

$$M_{,i}(\mathbf{x},k) = -\frac{a_3}{2}k^2\int_0^\infty (D\cdot A_{,i})\,\mathrm{d}u$$

$$M_{,ij}(\mathbf{x},k) = -\frac{a_3}{2}k^2\int_0^\infty \left(D\cdot A_{,ij}\right)\mathrm{d}u$$
$$M_{,ijm}(\mathbf{x},k) = -\frac{a_3}{2}k^2\int_0^\infty \left(D\cdot A_{,ijm}\right)\mathrm{d}u$$
$$M_{,ijmn}(\mathbf{x},k) = -\frac{a_3}{2}k^2\int_0^\infty \left(D\cdot A_{,ijmn}\right)\mathrm{d}u$$

where

$$A_{,\alpha} = B\cosh(Cx_3)I_1(B\rho)\frac{x_\alpha}{\rho}$$
$$A_{,\alpha\beta} = B\cosh(Cx_3)\frac{1}{2\rho^3}\left[B\rho[I_0(B\rho)+I_2(B\rho)]x_\alpha x_\beta + 2I_1(B\rho)\left(\rho^2\delta_{\alpha\beta}-x_\alpha x_\beta\right)\right]$$
$$\begin{aligned}A_{,\alpha\beta\gamma} = B\cosh(Cx_3)\cdot\Bigg\{&\left[-\frac{3B}{2\rho^4}x_\alpha x_\beta x_\gamma + \frac{B}{2\rho^2}\left(\delta_{\alpha\beta}x_\gamma+\delta_{\alpha\gamma}x_\beta+\delta_{\gamma\beta}x_\alpha\right)\right]I_0(B\rho)\\
&+\left[\frac{3}{\rho^5}x_\alpha x_\beta x_\gamma + \frac{3B^2}{4\rho^3}x_\alpha x_\beta x_\gamma - \frac{1}{\rho^3}\left(\delta_{\alpha\beta}x_\gamma+\delta_{\alpha\gamma}x_\beta+\delta_{\gamma\beta}x_\alpha\right)\right]I_1(B\rho)\\
&+\left[-\frac{3B}{2\rho^4}x_\alpha x_\beta x_\gamma + \frac{B}{2\rho^2}\left(\delta_{\alpha\beta}x_\gamma+\delta_{\alpha\gamma}x_\beta+\delta_{\gamma\beta}x_\alpha\right)\right]I_2(B\rho) + \left[\frac{B^2}{4\rho^3}x_\alpha x_\beta x_\gamma\right]I_3(B\rho)\Bigg\}\end{aligned}$$

$$\begin{aligned}A_{,\alpha\beta\gamma\lambda} = B\cosh(Cx_3)\cdot\Bigg\{&\frac{B}{\rho^6}\left[\frac{15}{2}x_\alpha x_\beta x_\gamma x_\lambda + \frac{3}{8}B^2\rho^2 x_\alpha x_\beta x_\gamma x_\lambda - \frac{3}{2}\rho^2 x_\lambda\left(\delta_{\alpha\beta}x_\gamma+\delta_{\alpha\gamma}x_\beta+\delta_{\gamma\beta}x_\alpha\right)\right.\\
&\left.-\frac{3}{2}\rho^2\left(\delta_{\alpha\lambda}x_\beta x_\gamma+\delta_{\beta\lambda}x_\alpha x_\gamma+\delta_{\gamma\lambda}x_\beta x_\alpha\right)+\frac{\rho^4}{2}\left(\delta_{\alpha\beta}\delta_{\gamma\lambda}+\delta_{\alpha\gamma}\delta_{\beta\lambda}+\delta_{\alpha\lambda}\delta_{\gamma\beta}\right)\right]I_0(B\rho)\\
&+\frac{1}{\rho^7}\left[-\frac{9}{2}B^2\rho^2 x_\alpha x_\beta x_\gamma x_\lambda - 15x_\alpha x_\beta x_\gamma x_\lambda + \frac{3}{4}B^2\rho^4 x_\lambda\left(\delta_{\alpha\beta}x_\gamma+\delta_{\alpha\gamma}x_\beta+\delta_{\gamma\beta}x_\alpha\right)\right.\\
&+3\rho^2 x_\lambda\left(\delta_{\alpha\beta}x_\gamma+\delta_{\alpha\gamma}x_\beta+\delta_{\gamma\beta}x_\alpha\right)+3\rho^2\left(\delta_{\alpha\lambda}x_\beta x_\gamma+\delta_{\beta\lambda}x_\alpha x_\gamma+\delta_{\gamma\lambda}x_\beta x_\alpha\right)\\
&\left.+\frac{3}{4}B^2\rho^4\left(\delta_{\alpha\lambda}x_\beta x_\gamma+\delta_{\beta\lambda}x_\alpha x_\gamma+\delta_{\gamma\lambda}x_\beta x_\alpha\right)-\rho^4\left(\delta_{\alpha\beta}\delta_{\gamma\lambda}+\delta_{\alpha\gamma}\delta_{\beta\lambda}+\delta_{\alpha\lambda}\delta_{\gamma\beta}\right)\right]I_1(B\rho)\\
&+\frac{B}{\rho^6}\left[\frac{15}{2}x_\alpha x_\beta x_\gamma x_\lambda + \frac{B^2\rho^2}{2}x_\alpha x_\beta x_\gamma x_\lambda - \frac{3}{2}\rho^2 x_\lambda\left(\delta_{\alpha\beta}x_\gamma+\delta_{\alpha\gamma}x_\beta+\delta_{\gamma\beta}x_\alpha\right)\right.\\
&\left.-\frac{3}{2}\rho^2\left(\delta_{\alpha\lambda}x_\beta x_\gamma+\delta_{\beta\lambda}x_\alpha x_\gamma+\delta_{\gamma\lambda}x_\beta x_\alpha\right)+\frac{\rho^4}{2}\left(\delta_{\alpha\beta}\delta_{\gamma\lambda}+\delta_{\alpha\gamma}\delta_{\beta\lambda}+\delta_{\alpha\lambda}\delta_{\gamma\beta}\right)\right]I_2(B\rho)\\
&+\frac{B^2}{\rho^5}\left[-\frac{3}{2}x_\alpha x_\beta x_\gamma x_\lambda + \frac{\rho^2}{4}x_\lambda\left(\delta_{\alpha\beta}x_\gamma+\delta_{\alpha\gamma}x_\beta+\delta_{\gamma\beta}x_\alpha\right)+\frac{\rho^2}{4}\left(\delta_{\alpha\lambda}x_\beta x_\gamma+\delta_{\beta\lambda}x_\alpha x_\gamma+\delta_{\gamma\lambda}x_\beta x_\alpha\right)\right]I_3(B\rho)\\
&+\frac{B^3}{\rho^4}\left[\frac{1}{8}x_\alpha x_\beta x_\gamma x_\lambda\right]I_4(B\rho)\Bigg\}\end{aligned}$$

The symbols α, β, γ, λ range from 1 to 2 and

$$A_{,3} = C\sinh(Cx_3)I_0(B\rho)$$
$$A_{,33} = C^2\cosh(Cx_3)I_0(B\rho),\quad A_{,\alpha 3} = (A_{,\alpha})_{,3}$$
$$A_{,333} = C^3\sinh(Cx_3)I_0(B\rho),\quad A_{,\alpha 33} = (A_{,\alpha})_{,33},\quad A_{,\alpha\beta 3} = (A_{,\alpha\beta})_{,3}$$
$$A_{,3333} = C^4\cosh(Cx_3)I_0(B\rho),\quad A_{,\alpha 333} = (A_{,\alpha})_{,333},\quad A_{,\alpha\beta 33} = (A_{,\alpha\beta})_{,33},\quad A_{,\alpha\beta\gamma 3} = (A_{,\alpha\beta\gamma})_{,3}$$

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