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Inclusion problem of microstretch continuum

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Abstract

A spherical inclusion in an infinite isotropic microstretch medium is examined in this paper. By means of Green's function technique, the analytical expressions of the Eshelby tensor for an isotropic microstretch medium are derived, and their volume averages over a spherical inclusion are obtained in an analytical and simple form. These results are useful to evaluate the effective property for a heterogeneous microstretch medium.

Keywords: Microstretch; Spherical inclusion; Eshelby tensor

1. Introduction

The concept of microcontinuum, proposed by Eringen [1], can take into account the microstructure effects while the theory itself is still a continuum formulation. The first grade microcontinuum consists a hierarchy of theories, namely, micropolar, microstretch and micromorphic, depending on how much microdegrees of freedom are incorporated. These high order continuum theories are considered to be potential tools to model the behavior of the material with a complicated microstructure. For example, in the case of a foam composite, when the size of the reinforced phase is comparable to the intrinsic length scale of the foam, in this situation, the microstructure of the foam must be taken into account to some degree, so a high order continuum model must be assigned for the foam matrix. The same remains true for nanocomposites, since the scale of the reinforced phase is so small, the surrounding matrix cannot be homogenized as a simple material (Cauchy medium), some intrinsic microstructures of the matrix must be considered in a proper continuum model.

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Microstretch [2,3] theory is a generalization of the micropolar theory, for such a material, a homogeneous stretch microdeformation is added to every particle, i.e., besides the translation and rigid rotation, each particle can have an independently breathe-like degree of freedom. Such a generalized media can catch more detailed information about the microdeformation inside a material point, which is more suitable for modeling the overall property of the foam matrix in the case of foam composites.

In classical Cauchy elasticity, the inclusion problem was first addressed by Eshelby [4]. The term *inclusion* originally appeared in Mura's monograph [5] represents a subdomain of a homogeneous material, which has been prescribed with certain distribution of eigenstrain, or stress-free strain. If such a domain has different properties from the surrounding matrix, it is called *inhomogeneity*. The above inclusion theory forms the basis for micromechanics to predict the overall properties of heterogeneous materials, i.e., composites, materials with defects. Recently by using the Green's function technique, Cheng and He [6,7] extended the inclusion problem to micropolar elasticity. In their work a similar term *eigentorsion* was introduced in addition to the classical eigenstrain, and consequently four Eshelby tensors are obtained for a micropolar material. Even for the simplest shape (for example a sphere), the Eshelby tensors are not uniform inside a spherical inclusion, and to the present only the Eshelby tensors for spherical and infinite cylindrical inclusions are analytically derived. Recently Liu and Hu [8] make use of these results, and obtain analytically the average Eshelby tensors over a spherical region for micropolar material. With these average Eshelby tensors, they further generalize the classical micromechanics for a heterogeneous Cauchy medium to a micropolar composite, the influence of the particle's size on the overall elastic–plastic behavior of composite materials is correctly predicted.

Encouraged by these results, in the paper, we will examine the inclusion problem for a microstretch continuum, this is the first step towards the potential application of microstretch theory to predict the overall behavior of heterogeneous materials. The microstretch version of inclusion problem results in new Eshelby tensors due to the new strain measure introduced by the theory, of cause when the microstretch effect is neglected, the results for a micropolar counterpart must be recovered. Our work is limited to an isotropic material and a spherical inclusion, to facilitate further application for micromechanics, the analytical expressions for the average Eshelby tensors over a spherical region in a microstretch medium will also be given. The paper is arranged as the follows, In Section 2, a brief recall and the definition of an eigenstrain problem for microstretch theory will be outlined. In Section 3, the fundamental solution will be completed and the general expression of the field quantities caused by the eigenstrain will be formulated; and the solution of an inclusion problem will be presented in Section 4; finally, the average Eshelby tensors over a spherical inclusion will be performed and the analytical expressions will be given. In most case indices notation for a tensor (or vector) is adopted in this paper, except some vector representations appear in bold letter as used for convenience.

2. Basic equations and symbolic notations

We denote u_i and ϕ_i as the displacement and microrotation of a material point respectively, and θ represents the microstretch (or contraction) to that point. The strain measures of a microstretch media consist of these quantities themselves and their space gradients, the three sets of governing relations for an isotropic microstretch continuum are:

geometrical relations:

$$\varepsilon_{ji} = u_{i,j} + e_{ijk}\phi_k \quad k_{ji} = \phi_{i,j} \quad \zeta_i = \theta_{,i} \tag{1}$$

where e_{ijk} is the third order permutation tensor, a comma preceded by a subscript means space derivative. The stress quantities, as the strain measure's dual part, must satisfy the balance equations:

balance equations:

$$t_{ji,j} + f_i = 0 \quad m_{ji,j} + e_{ikl}t_{kl} + l_i = 0 \quad p_{k,k} - s + l = 0 \tag{2}$$

where t_{ji}, m_{ji} are asymmetric stress and couple-stress, p_k and s is new stress quantities additional to micropolar theory, which are thermodynamically conjugate to $\theta_{,i}$ and θ. f_i, l_i and l are body loads which make balance in the three equations. Finally, for an isotropic microstretch solid, the stress and strain are related by the following constitutive equation:

constitutive equations:

$$t_{ji} = C_{jikl}\varepsilon_{kl} + \delta_{ji}\lambda_0\theta \quad m_{ji} = D_{jikl}k_{kl} \quad p_i = \eta\theta_{,i} \quad s = \lambda_0\varepsilon_{kk} + b\theta \tag{3}$$

where C_{jikl}, D_{jikl} is the isotropic modulus tensors of the following form:

$$\begin{aligned} C_{jikl} &= \lambda\delta_{ji}\delta_{kl} + \left(\mu + \frac{\kappa}{2}\right)\delta_{jk}\delta_{il} + \left(\mu - \frac{\kappa}{2}\right)\delta_{jl}\delta_{ik} \\ D_{jikl} &= \alpha\delta_{ji}\delta_{kl} + \gamma\delta_{jk}\delta_{il} + \beta\delta_{jl}\delta_{ik} \end{aligned} \tag{4}$$

There are in total nine independent material constants, λ, μ, κ, α, β, γ are micropolar constants, and λ_0, η and b are new constants due to the generalization to microstretch theory. For a well-posed boundary value problem, the following boundary conditions must be provided:

$$\begin{aligned} &t_{ji}n_j = \bar{t}_i \quad m_{ji}n_j = \bar{m}_i \quad p_k n_k = \bar{p} \quad \text{on } S^\sigma \\ &u_i = \bar{u}_i \quad \phi_i = \bar{\phi}_i \quad \theta = \bar{\theta} \quad \text{on } S^u \end{aligned} \tag{5}$$

where n_j is the outer normal of the boundary.

Eigenstrain, or stress-free strain, is usually used to simulate the thermal expansion, phase transformation, initial strains, plastic strain or misfit strains. The incompatibility of eigenstrain will result in a self-equilibrium stress field in a material free from any external load. Here this concept from a classical Cauchy media is generalized for a microstrech continuum as an eigenstrain $\varepsilon^+_{ji}(\mathbf{x})$, an eigentorsion $k^+_{ji}(\mathbf{x})$, an eigenmicrostretch-gradient $\zeta^+_k(\mathbf{x})$ and an eigenmicrostretch $\theta^+(\mathbf{x})$, all these eigenvariables are called eigendeformation in the following.

Consider an infinitely extended microstretch elastic body with a distribution of eigendeformation

$$\varepsilon_{ji}^{+}(\mathbf{x}) \quad k_{ji}^{+}(\mathbf{x}) \quad \zeta_{k}^{+}(\mathbf{x}) \quad \text{and} \quad \theta^{+}(\mathbf{x})$$

the constitutive equation (3) must be modified due to the introduction of such nonelastic deformation

$$\begin{aligned} &t_{ji} = C_{jikl}(\varepsilon_{kl} - \varepsilon_{kl}^{+}) + \delta_{ji}\lambda_0(\theta - \theta^{+}) \quad m_{ji} = D_{jikl}(k_{kl} - k_{kl}^{+}) \\ &p_i = \eta(\theta_{,i} - \zeta_i^{+}) \quad s = \lambda_0(\varepsilon_{kk} - \varepsilon_{kk}^{+}) + b(\theta - \theta^{+}) \end{aligned} \tag{6}$$

In absence of body force, by substitution of the geometrical equation (1) into the constitutive relation (6) and then into the balance equations (2), we obtain the following governing equations:

$$\begin{aligned} &C_{jikl}u_{l,kj} + \kappa e_{ikl}\phi_{l,k} + \lambda_0\theta_{,i} + f_i^{+} = 0 \\ &D_{jikl}\phi_{l,kj} + \kappa e_{ikl}u_{l,k} - 2\kappa\phi_i + l_i^{+} = 0 \\ &\eta\theta_{,kk} - \lambda_0 u_{r,r} - b\theta + l^{+} = 0 \end{aligned} \tag{7}$$

where

$$f_i^{+} = -(C_{jikl}\varepsilon_{kl,j}^{+} + \lambda_0\theta_{,i}^{+}) \quad l_i^{+} = -(\kappa e_{ikl}\varepsilon_{kl}^{+} + D_{jikl}k_{kl,j}^{+}) \quad l^{+} = \lambda_0\varepsilon_{kk}^{+} + b\theta^{+} - \eta\zeta_{k,k}^{+} \tag{8}$$

Here the role of the introduced eigendeformation is transformed as the equivalent body forces.

3. Solution of elastic field

Determination of u_i, ϕ_i and θ can apply standard Green's function technique. The Green's functions of a microstretch media are not explicitly at hand, However Eringen [1] has indicated a way to obtain the fundamental solution for a microstretch media. We give directly the expressions of Green's function here and the detail derivation is explained in Appendix A. Assuming in an infinitely extended body there exist impulse body loads at the position $\mathbf{x}'$

$$\mathbf{f} = \mathbf{F}\delta(\mathbf{x} - \mathbf{x}') \quad \mathbf{l} = \mathbf{L}\delta(\mathbf{x} - \mathbf{x}') \quad l = L\delta(\mathbf{x} - \mathbf{x}') \tag{9}$$

Then the fundamental solution of microstretch theory can be summarized in the following expressions:

$$\begin{aligned} &u_i = \left(G_{ij}^{1} + G_{ij}^{\text{Stretch}}\right)F_j + G_{ij}^{2}L_j + G_iL \\ &\phi_i = H_{ij}^{1}F_j + H_{ij}^{2}L_j \\ &\theta = \Theta_jF_j + \Theta L \end{aligned} \tag{10}$$

where G^1_{ij}, $G^2_{ij} = H^1_{ij}$ and H^2_{ij} are the Green's functions for micropolar theory, one can refer to Sandru [9] for their detail expressions, and G^{Stretch}_{ij}, G_i, Θ_i and Θ are the additional Green's functions due to the incorporation of microstretch effect, they read

$$
\begin{aligned}
G^{\text{Stretch}}_{ij} &= \frac{bp^2-\eta}{4\pi\eta(\lambda+2\mu)}\left\{\frac{1}{2}\frac{\partial^2 r}{\partial x_i\partial x_j} + p^2\frac{\partial^2}{\partial x_i\partial x_j}\left(\frac{1-\mathrm{e}^{-r/p}}{r}\right)\right\} \\
G_i = -\Theta_i &= \frac{\lambda_0 p^2}{4\pi\eta(\lambda+2\mu)}\frac{\partial}{\partial x_i}\left(\frac{1-\mathrm{e}^{-r/p}}{r}\right) \\
\Theta &= \frac{1}{4\pi\eta}\frac{\mathrm{e}^{-r/p}}{r}
\end{aligned}
\tag{11}
$$

where $p = \sqrt{\frac{\eta(\lambda+2\mu)}{(\lambda+2\mu)b-\lambda_0^2}}$ has a dimension of length, and $r = |\mathbf{x}-\mathbf{x}'|$.

With the help of the obtained fundamental solutions, the whole displacement fields u_i, ϕ_i and θ under the previously prescribed eigendeformation can be readily obtained. To do this we firstly use the work reciprocal theorem of a microstretch media

$$
\int_V \left(f_i u'_i + l_i\phi'_i + l\theta'\right)\mathrm{d}\mathbf{x} = \int_V \left(f'_i u_i + l'_i\phi_i + l'\theta\right)\mathrm{d}\mathbf{x} \tag{12}
$$

where the quantities with and without a prime are two distinct independent sets of the load and the resulted displacement fields. By taking respectively,

$$
\begin{aligned}
\left(f'_i, l'_i, l', u'_i, \phi'_i, \theta'\right) &= \left(\delta_{ik}\delta(\mathbf{x}-\mathbf{x}'), 0, 0, G^1_{ik} + G^{\text{Stretch}}_{ik}, H^1_{ik}, \Theta_k\right) \\
&= \left(0, \delta_{ik}\delta(\mathbf{x}-\mathbf{x}'), 0, G^2_{ik}, H^2_{ik}, 0\right) = \left(0, 0, \delta(\mathbf{x}-\mathbf{x}'), G_i, 0, \Theta\right)
\end{aligned}
\tag{13}
$$

we have the general expressions of u_i, ϕ_i and θ for an infinitely extended body under the body loads $f_i(\mathbf{x})$, $l_i(\mathbf{x})$, $l(\mathbf{x})$

$$
\begin{aligned}
u_k(\mathbf{x}) &= \int_V \left\{f_i(\mathbf{x}')[G^1_{ik}(\mathbf{x}-\mathbf{x}') + G^{\text{Stretch}}_{ik}(\mathbf{x}-\mathbf{x}')] + l_i(\mathbf{x}')H^1_{ik}(\mathbf{x}-\mathbf{x}') - l(\mathbf{x}')\Theta_k(\mathbf{x}-\mathbf{x}')\right\}\mathrm{d}\mathbf{x}' \\
\phi_k(\mathbf{x}) &= \int_V \left\{f_i(\mathbf{x}')G^2_{ik}(\mathbf{x}-\mathbf{x}') + l_i(\mathbf{x}')H^2_{ik}(\mathbf{x}-\mathbf{x}')\right\}\mathrm{d}\mathbf{x}' \\
\theta(\mathbf{x}) &= \int_V \left\{-f_i(\mathbf{x}')G_i(\mathbf{x}-\mathbf{x}') + l(\mathbf{x}')\Theta(\mathbf{x}-\mathbf{x}')\right\}\mathrm{d}\mathbf{x}'
\end{aligned}
\tag{14}
$$

By substituting the expression of body loads defined by Eq. (8) into Eq. (14), and integrating by parts, the final solutions for the local displacement fields due to the prescribed eigendeformation read

$$
\begin{aligned}
u_n(\mathbf{x}) &= -\int_V \Big\{ C_{jikl}\varepsilon^+_{kl}(\mathbf{x})\Big(G^1_{in,j} + G^{\text{Stretch}}_{in,j}\Big) + \kappa e_{ikl}\varepsilon^+_{kl}(\mathbf{x})H^1_{in} + D_{jikl}k^+_{kl}(\mathbf{x})H^1_{in,j} \\
&\quad + \lambda_0\theta^+(\mathbf{x})\Big(G^1_{in,i} + G^{\text{Stretch}}_{in,i}\Big) + \lambda_0\varepsilon^+_{rr}(\mathbf{x})\Theta_n + b\theta^+(\mathbf{x})\Theta_n - \eta\zeta^+_r(\mathbf{x})\Theta_{n,r}\Big\}\mathrm{d}\mathbf{x}' \\
\phi_n(\mathbf{x}) &= -\int_V \Big\{ C_{jikl}\varepsilon^+_{kl}(\mathbf{x})G^2_{in,j} + \kappa e_{ikl}\varepsilon^+_{kl}(\mathbf{x})H^2_{in} + D_{jikl}k^+_{kl}(\mathbf{x})H^2_{in,j}\Big\}\mathrm{d}\mathbf{x}' \\
\theta(\mathbf{x}) &= \int_V \big\{ C_{jikl}\varepsilon^+_{kl}(\mathbf{x})G_{i,j} + \lambda_0\theta^+(\mathbf{x})G_{i,i} + \lambda_0\varepsilon^+_{rr}(\mathbf{x})\Theta + b\theta^+(\mathbf{x})\Theta - \eta\zeta^+_r(\mathbf{x})\Theta_{,r}\big\}\mathrm{d}\mathbf{x}'
\end{aligned}
\tag{15}
$$

These general expressions, together with aid of Eqs. (11), (1), (6), provide the complete local elastic displacement, strain and stress fields due to any eigendeformation. In the next section, this general result will be applied to the inclusion problem.

4. Inclusion problem

Consider an infinite extended elastic body, in a spherical subdomain Ω, there is a *uniform* eigendeformation (ε^+_{ij}, k^+_{ij}, ζ^+_i, θ^+) and this eigendeformation is zero outside of Ω.

Eq. (15) can be rearranged in the following form:

$$
\begin{aligned}
u_n(\mathbf{x}) &= u^{\text{polar}}_n(\mathbf{x}) + I^{\text{Stretch}}_{nkl}(\mathbf{x})\varepsilon^+_{kl} + I^{\text{Stretch}}_{nk}(\mathbf{x})\zeta^+_k + I^{\text{Stretch}}_n(\mathbf{x})\theta^+ \\
\phi_n(\mathbf{x}) &= \phi^{\text{polar}}_n(\mathbf{x}) \\
\theta(\mathbf{x}) &= J^{\text{Stretch}}_{kl}(\mathbf{x})\varepsilon^+_{kl} + J^{\text{Stretch}}_k(\mathbf{x})\zeta^+_k + J^{\text{Stretch}}(\mathbf{x})\theta^+
\end{aligned}
\tag{16}
$$

where $u^{\text{polar}}_n(\mathbf{x})$ and $\phi^{\text{polar}}_n(\mathbf{x})$ are just the micropolar elastic fields of the corresponding inclusion problem given by Cheng and He [6,7]. We just give in the following the other six coefficient functions with a superscript *Stretch* indicating the relevant quantities due to the microstretch effect. From above equations, we note that the microrotation ϕ_n is independent of the microstretch effect, it just remains the same form as that in micropolar theory. The other six coefficient functions are

$$
\begin{aligned}
I^{\text{Stretch}}_{nkl} &= -C_{jikl}\int_\Omega G^{\text{Stretch}}_{in,j}\,\mathrm{d}V - \lambda_0\delta_{kl}\int_\Omega \Theta_n\,\mathrm{d}V \\
I^{\text{Stretch}}_{nk} &= \eta\int_\Omega \Theta_{n,k}\,\mathrm{d}V \\
I^{\text{Stretch}}_{n} &= -\lambda_0\int_\Omega \Big(G^1_{in,i} + G^{\text{Stretch}}_{in,i}\Big)\mathrm{d}V - b\int_\Omega \Theta_n\,\mathrm{d}V \\
J^{\text{Stretch}}_{kl} &= C_{jikl}\int_\Omega G_{i,j}\,\mathrm{d}V + \lambda_0\delta_{kl}\int_\Omega \Theta\,\mathrm{d}V \\
J^{\text{Stretch}}_{k} &= -\eta\int_\Omega \Theta_{,k}\,\mathrm{d}V \\
J^{\text{Stretch}} &= \lambda_0\int_\Omega G_{i,i}\,\mathrm{d}V + b\int_\Omega \Theta\,\mathrm{d}V
\end{aligned}
\tag{17}
$$

With help of Eqs. (11) and (4), we obtain finally

$$
\begin{aligned}
I_{nkl}^{\text{Stretch}} &= \left(\frac{\lambda}{\lambda+2\mu}-\frac{b\lambda-\lambda_0^2}{A}\right)\delta_{kl}\phi_{,n}(\mathbf{x})+\left(\frac{\mu}{\lambda+2\mu}-\frac{b\mu}{A}\right)\psi_{,nkl}(\mathbf{x})-\frac{\lambda_0^2}{A}\delta_{kl}M_{,n}(\mathbf{x})\\
&\quad+\left(\frac{b\eta(\lambda+2\mu)}{A^2}-\frac{\eta}{A}\right)[\lambda\delta_{kl}M_{,iin}(\mathbf{x},p)+2\mu M_{,nkl}(\mathbf{x},p)-2\mu\phi_{,nkl}(\mathbf{x})]\\
I_{nk}^{\text{Stretch}} &= -\frac{\eta\lambda_0}{A}[\phi_{,nk}(\mathbf{x})-M_{,nk}(\mathbf{x},p)]\\
I_{n}^{\text{Stretch}} &= -\lambda_0\left[Bh^2M(\mathbf{x},h)_{,nii}-BM(\mathbf{x},h)_{,n}+\frac{1}{\lambda+2\mu}\phi(\mathbf{x})_{,n}\right]\\
&\quad-\lambda_0\left(\frac{b}{A}-\frac{1}{\lambda+2\mu}\right)\left\{\phi(\mathbf{x})_{,n}-p^2M(\mathbf{x},p)_{,nii}\right\}+\frac{b\lambda_0}{A}[\phi(\mathbf{x})_{,n}-M(\mathbf{x},p)_{,n}]\\
J_{kl}^{\text{Stretch}} &= \frac{\lambda_0}{\eta}\delta_{kl}M(\mathbf{x},p)+\frac{\lambda_0}{A}[2\mu\phi_{,kl}(\mathbf{x})-2\mu M_{,kl}(\mathbf{x},p)-\lambda\delta_{kl}M_{,ii}(\mathbf{x},p)]\\
J_{k}^{\text{Stretch}} &= -M_{,k}(\mathbf{x},p)\\
J^{\text{Stretch}} &= \frac{b}{\eta}M(\mathbf{x},p)-\frac{\lambda_0^2}{A}M(\mathbf{x},p)_{,ii}
\end{aligned}
\tag{18}
$$

where $A=(\lambda+2\mu)b-\lambda_0^2$, $B=\kappa/[\mu(2\mu+\kappa)]$, $h^2=(2\mu+\kappa)\gamma/4\mu\kappa$ and

$$
\psi(\mathbf{x})=\frac{1}{4\pi}\int_\Omega r\mathrm{d}\mathbf{x}' \quad \phi(\mathbf{x})=\frac{1}{4\pi}\int_\Omega\frac{1}{r}\mathrm{d}\mathbf{x}' \quad M(\mathbf{x},k)=\frac{1}{4\pi}\int_\Omega\frac{\mathrm{e}^{-r/k}}{r}\mathrm{d}\mathbf{x}' \tag{19}
$$

The analytical solution of the above three integrations is hard to obtain for a general shape of inclusion, especially for the third one. For a spherical inclusion, the analytical expression of $M(\mathbf{x},k)$ is given by Cheng and He [6], and for the other two integrations in (19), we refer the readers to the Ref. [5]. In Appendix B we explicitly list these integrals for spherical domain.

The associated strain measures can be obtained without any difficulty through Eq. (1)

$$
\begin{aligned}
\varepsilon_{mn}(\mathbf{x}) &= \left[K_{mnkl}(\mathbf{x})+K_{mnkl}^{\text{Stretch}}(\mathbf{x})\right]\varepsilon_{kl}^{+}+L_{mnkl}(\mathbf{x})k_{kl}^{+}+N_{mnk}^{\text{Stretch}}(\mathbf{x})\zeta_k^{+}+T_{mn}^{\text{Stretch}}(\mathbf{x})\theta^{+}\\
k_{mn}(\mathbf{x}) &= \widehat{K}_{mnkl}(\mathbf{x})\varepsilon_{kl}^{+}+\widehat{L}_{mnkl}(\mathbf{x})k_{kl}^{+}\\
\zeta_n(\mathbf{x}) &= K_{nkl}^{\text{Stretch}}(\mathbf{x})\varepsilon_{kl}^{+}+N_{nk}^{\text{Stretch}}(\mathbf{x})\zeta_k^{+}+T_{n}^{\text{Stretch}}(\mathbf{x})\theta^{+}
\end{aligned}
\tag{20}
$$

where K_{mnji} L_{mnji} $\widehat{K}_{mnji}$ and $\widehat{L}_{mnji}$ are the four Eshelby tensors for a micropolar medium (see Ref. [6]), and

$$
\begin{aligned}
&K_{mnkl}^{\text{Stretch}}(\mathbf{x})=I_{nkl,m}^{\text{Stretch}}(\mathbf{x}) \quad K_{nkl}^{\text{Stretch}}(\mathbf{x})=J_{kl,n}^{\text{Stretch}}(\mathbf{x})\\
&N_{mnk}^{\text{Stretch}}(\mathbf{x})=I_{nk,m}^{\text{Stretch}}(\mathbf{x}) \quad N_{nk}^{\text{Stretch}}(\mathbf{x})=J_{k,n}^{\text{Stretch}}(\mathbf{x})\\
&T_{mn}^{\text{Stretch}}(\mathbf{x})=I_{n,m}^{\text{Stretch}}(\mathbf{x}) \quad T_{n}^{\text{Stretch}}(\mathbf{x})=J_{,n}^{\text{Stretch}}(\mathbf{x})
\end{aligned}
\tag{21}
$$

These are the additional Eshelby tensors due to the microstretch effect.

With help of the analytical expressions for the integrals (19), all the Eshelby tensors for a microstretch continuum can be obtained through Eqs. (18)–(21). As for the micropolar continuum, the derived Eshelby tensors are position dependent even inside of a spherical inclusion. In order to be used for predicting the overall property of a heterogeneous microstretch medium, we will give in the following the analytical expressions for the average Eshelby tensors. Averaging both side of Eq. (20) and the third one of Eq. (16) over a spherical region, we get

$$\begin{aligned}
\langle \varepsilon_{mn} \rangle_\Omega &= [\langle K_{mnkl} \rangle_\Omega + \langle K_{mnkl}^{\text{Stretch}} \rangle_\Omega] \varepsilon_{kl}^{+} + \langle L_{mnkl} \rangle_\Omega k_{kl}^{+} + \langle N_{mnk}^{\text{Stretch}} \rangle_\Omega \zeta_k^{+} + \langle T_{mn}^{\text{Stretch}} \rangle_\Omega \theta^{+} \\
\langle k_{mn} \rangle_\Omega &= \langle \widehat{K}_{mnkl} \rangle_\Omega \varepsilon_{kl}^{+} + \langle \widehat{L}_{mnkl} \rangle_\Omega k_{kl}^{+} \\
\langle \zeta_n \rangle_\Omega &= \langle K_{nkl}^{\text{Stretch}} \rangle_\Omega \varepsilon_{kl}^{+} + \langle N_{nk}^{\text{Stretch}} \rangle_\Omega \zeta_k^{+} + \langle T_n^{\text{Stretch}} \rangle_\Omega \theta^{+} \\
\langle \theta \rangle_\Omega &= \langle J_{kl}^{\text{Stretch}} \rangle_\Omega \varepsilon_{kl}^{+} + \langle J_k^{\text{Stretch}} \rangle_\Omega \zeta_k^{+} + \langle J^{\text{Stretch}} \rangle_\Omega \theta^{+}
\end{aligned} \tag{22}$$

Here $\langle * \rangle_\Omega = \int_\Omega *(\mathbf{x}) \mathrm{d}\mathbf{x}$, representing the volume average with respect to the inclusion domain. For a spherical inclusion and an isotropic microstretch material, the average Eshelby tensors over the inclusion are isotropic and have the following form:

$$\begin{aligned}
\langle K_{mnkl} \rangle_\Omega &= K_1 \delta_{mn}\delta_{kl} + (K_2 + K_3)\delta_{mk}\delta_{nl} + (K_2 - K_3)\delta_{ml}\delta_{nk} \\
\langle K_{mnkl}^{\text{Stretch}} \rangle_\Omega &= K_1^{\text{Stretch}} \delta_{mn}\delta_{kl} + (K_2^{\text{Stretch}} + K_3^{\text{Stretch}})\delta_{mk}\delta_{nl} + (K_2^{\text{Stretch}} - K_3^{\text{Stretch}})\delta_{ml}\delta_{nk} \\
\langle \widehat{L}_{mnkl} \rangle_\Omega &= \widehat{L}_1 \delta_{mn}\delta_{kl} + (\widehat{L}_2 + \widehat{L}_3)\delta_{mk}\delta_{nl} + (\widehat{L}_2 - \widehat{L}_3)\delta_{ml}\delta_{nk} \\
\langle N_{nk}^{\text{Stretch}} \rangle_\Omega &= \delta_{nk} N^{\text{Stretch}} \\
\langle T_{mn}^{\text{Stretch}} \rangle_\Omega &= \delta_{mn} T^{\text{Stretch}} \\
\langle \widehat{K}_{mnkl} \rangle_\Omega &= \langle L_{mnkl} \rangle_\Omega = \langle K_{nkl}^{\text{Stretch}} \rangle_\Omega = 0 \\
\langle N_{mnk}^{\text{Stretch}} \rangle_\Omega &= \langle T_n^{\text{Stretch}} \rangle_\Omega = \langle J_k^{\text{Stretch}} \rangle_\Omega = 0
\end{aligned} \tag{23}$$

We denote that tensors with odd order eventually averages to zero. After a lengthy mathematical manipulation, we get

$$\begin{aligned}
K_1 &= \frac{3\lambda - 2\mu}{15(\lambda + 2\mu)} + \frac{2h(a+h)\kappa}{5a^3(\kappa + 2\mu)} \Gamma(h) \\
K_2 &= \frac{3\lambda + 8\mu}{15(\lambda + 2\mu)} - \frac{3h(a+h)\kappa}{5a^3(\kappa + 2\mu)} \Gamma(h) \\
K_3 &= \frac{4\kappa + 3\mu}{6\mu} - \frac{2h(a+h)(\mu+\kappa)^2}{a^3\mu(\kappa + 2\mu)} \Gamma(h) - \frac{g(a+g)}{2a^3} \Gamma(g) \\
\widehat{L}_1 &= \frac{(a+g)(5\alpha + \beta + \gamma)\mu}{10a^3 g\kappa\mu} \Gamma(g) - \frac{(h+a)(\beta+\gamma)(\kappa+2\mu)}{20a^3 h\kappa\mu} \Gamma(h) \\
\widehat{L}_2 &= \frac{(a+g)(\beta+\gamma)\mu}{10a^3 g\kappa\mu} \Gamma(g) + \frac{3(a+h)(\beta+\gamma)(\kappa+2\mu)}{40a^3 h\kappa\mu} \Gamma(h)
\end{aligned}$$

$$
\begin{aligned}
\widehat{L}_3 &= \frac{(a+h)(\gamma-\beta)(\kappa+2\mu)}{8a^3h\kappa\mu}\Gamma(h) \\
K_1^{\text{Stretch}} &= \frac{8(a+p)\mu\eta\lambda_0^2}{5a^3A^2p}\Gamma(p) - \frac{8\mu\lambda_0^2}{15A(\lambda+2\mu)} \\
K_2^{\text{Stretch}} &= -\frac{2(a+p)\mu\eta\lambda_0^2}{5a^3A^2p}\Gamma(p) + \frac{2\mu\lambda_0^2}{15A(\lambda+2\mu)} \\
K_3^{\text{Stretch}} &= 0 \\
N^{\text{Stretch}} &= \frac{p(a+p)}{a^3}\Gamma(p) \\
T^{\text{Stretch}} &= \frac{p(a+p)\lambda_0}{a^3(\lambda+2\mu)}\Gamma(p) \\
\langle J_{kl}^{\text{Stretch}}\rangle_\Omega &= \frac{\lambda_0(3\lambda+4\mu)}{3A} - \frac{4p(a+p)\mu\lambda_0}{a^3A}\Gamma(p) \\
\langle J^{\text{Stretch}}\rangle_\Omega &= \frac{p^2b}{\eta} - \frac{3p(a+p)}{a^3}\Gamma(p)
\end{aligned}
\tag{24}
$$

where

$$
\Gamma(y) = \mathrm{e}^{-a/y}\left[a\,\mathrm{Cosh}\frac{a}{y} - y\,\mathrm{Sinh}\frac{a}{y}\right] \tag{25}
$$

$$
\begin{aligned}
h &= \sqrt{\gamma(2\mu+\kappa)/4\mu\kappa} \\
g &= \sqrt{(\alpha+\beta+\gamma)/2\kappa}
\end{aligned}
\tag{26}
$$

notice that h, g and p can be regarded as the characteristic length of a microstretch material; a is the radius of the spherical inclusion.

5. Conclusion

We therefore derive the analytical expressions of the Eshelby tensors for a spherical inclusion in an isotropic microstretch medium, the volume averages of the derived Eshelby tensors over a spherical inclusion are also obtained in a simple and analytical form. These results can be used to evaluate the average local fields in a spherical inhomogeneity for a microstretch material, which is essential for predicting the overall property for a heterogeneous microstretch medium.

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Appendix A

The fundamental solution of a microstretch continuum has been discussed by Eringen [1], here we complete the final small step left. The balance equations read with the vector presentation

$$\begin{aligned}&\lambda_0\nabla\theta+(\lambda+2\mu)\nabla\nabla\cdot\mathbf{u}-(\mu+\kappa/2)\nabla\times\nabla\times\mathbf{u}+\kappa\nabla\times\boldsymbol{\varphi}+\mathbf{f}=0\\&(\alpha+\beta+\gamma)\nabla\nabla\cdot\boldsymbol{\varphi}-\gamma\nabla\times\nabla\times\boldsymbol{\varphi}+\kappa\nabla\times\mathbf{u}-2\kappa\boldsymbol{\varphi}+\mathbf{l}=0\\&\eta\nabla^2\theta-b\theta-\lambda_0\nabla\cdot\mathbf{u}+l=0\end{aligned}\tag{A.1}$$

where the impulse body loads take the form defined by Eq. (9). Green's functions come out from solution of the previous problem for an infinite extended microstretch elastic solid.

The decomposition of the displacement and rotation into scalar and vector potentials in a micropolar media is still valid in microstretch context, among various decomposition, we make use of Sandru's [9] representation

$$\begin{aligned}&\mathbf{u}=\nabla\Lambda_0+\nabla\times(\Diamond_4\boldsymbol{\Lambda})-\kappa\nabla\times(\nabla\times\boldsymbol{\Lambda}^*)\quad\text{where }\nabla\cdot\boldsymbol{\Lambda}=0\\&\boldsymbol{\varphi}=\nabla\Lambda_0^*-\kappa\nabla\times(\nabla\times\Lambda)+\nabla\times(\Diamond_2\boldsymbol{\Lambda}^*)\quad\text{where }\nabla\cdot\boldsymbol{\Lambda}^*=0\end{aligned}\tag{A.2a}$$

$$\begin{aligned}&\mathbf{f}=\nabla\tau+\nabla\times\Pi\quad\text{where }\nabla\cdot\Pi=0\\&\mathbf{l}=\nabla\tau'+\nabla\times\Pi'\quad\text{where }\nabla\cdot\Pi'=0\end{aligned}\tag{A.2b}$$

and where

$$\begin{aligned}&\Diamond_1=(\lambda+2\mu)\nabla^2\quad\Diamond_2=(\mu+\kappa/2)\nabla^2\\&\Diamond_3=(\alpha+\beta+\gamma)\nabla^2-2\kappa\quad\Diamond_4=\gamma\nabla^2-2\kappa\end{aligned}\tag{A.3}$$

By such manipulation, the balance equations (A.1) are transformed to the following five equations:

$$(\Diamond_2\Diamond_4+\kappa^2\nabla^2)\boldsymbol{\Lambda}^*+\Pi'=0\tag{A.4a}$$

$$(\Diamond_2\Diamond_4+\kappa^2\nabla^2)\boldsymbol{\Lambda}+\Pi=0\tag{A.4b}$$

$$\Diamond_3\Lambda_0^*+\tau'=0\tag{A.4c}$$

$$\lambda_0\theta+\Diamond_1\Lambda_0+\tau=0\tag{A.4d}$$

$$\eta\nabla^2\theta-b\theta-\lambda_0\nabla^2\Lambda_0+l=0\tag{A.4e}$$

Here, fortunately, we find that the governing equations for a microstretch media are quite similar to the micropolar version after the decomposition by the potentials. In fact, the first three equations are uncoupled and identical to those in a micropolar media; only the last two coupled equations with the unknown variables θ (microdilatation) and Λ_0 (scalar potential of displacement) need to be solved here.

Eqs. (A.2b) and (9) lead to

$$\tau = -\frac{1}{4\pi}\mathbf{F}\cdot\nabla\left(\frac{1}{r}\right) \quad \text{where } r = \left|\mathbf{x}-\mathbf{x}'\right| \tag{A.5}$$

using Fourier transformation technique, we obtain the following solution:

$$\begin{aligned} \Lambda_0 &= \frac{p^2\mathbf{F}\cdot}{4\pi\eta(\lambda+2\mu)}\left\{\frac{b}{2}\nabla r - (\eta - bp^2)\nabla\frac{1-\mathrm{e}^{-r/p}}{r}\right\} + \frac{\lambda_0 p^2 L}{4\pi\eta(\lambda+2\mu)}\frac{1-\mathrm{e}^{-r/p}}{r} \\ \theta &= \frac{-\lambda_0 p^2\mathbf{F}\cdot}{4\pi\eta(\lambda+2\mu)}\nabla\frac{1-\mathrm{e}^{-r/p}}{r} + \frac{L}{4\pi\eta}\frac{\mathrm{e}^{-r/p}}{r} \end{aligned} \tag{A.6}$$

Substitution of the above expressions and the micropolar solutions of Λ_0^*, $\mathbf{\Lambda}$ and $\mathbf{\Lambda}^*$ (see for example Ref. [8]) into (A.2a), after some manipulation, we obtain Eqs. (10) and (11) in the text.

Appendix B

The integrals appeared in Eq. (19) for a spherical inclusion are evaluated by Cheng and He [6], the results are

$$\psi(\mathbf{x}) = \begin{cases} -\dfrac{1}{60}(x^4 - 10a^2x^2 - 15a^4) & \mathbf{x}\in\Omega \\ \dfrac{a^3}{15}\left(5x + \dfrac{a^2}{x}\right) & \mathbf{x}\notin\Omega \end{cases}$$

$$\phi(\mathbf{x}) = \begin{cases} -\dfrac{1}{6}(x^2 - 3a^2) & \mathbf{x}\in\Omega \\ \dfrac{a^3}{3x} & \mathbf{x}\notin\Omega \end{cases}$$

$$M(\mathbf{x},k) = \begin{cases} k^2 - k^2(k+a)\mathrm{e}^{-a/k}\dfrac{\mathrm{Sinh}(x/k)}{x} & \mathbf{x}\in\Omega \\ k^2\left(a\,\mathrm{Cosh}\,\frac{a}{k} - k\,\mathrm{Sinh}\,\frac{a}{k}\right)\dfrac{\mathrm{e}^{-x/k}}{x} & \mathbf{x}\notin\Omega \end{cases}$$

where $x = |\mathbf{x}|$.

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