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Influence of residual stress on the elastic-plastic deformation of composites with two- or three-dimensional randomly oriented inclusions

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Summary. Based on the secant moduli framework with second-order moment of stress and the elastic micro-mechanical method proposed by Ponte Castaneda and Willis [8], a micro-mechanical method is proposed to consider the effects of thermal residual stress on the elastic-plastic deformation of composites. The micro-structural parameters like the volume fraction, shape, orientation, and distribution of inclusions are taken into account, and their influences on the macroscopic properties of the composites in the presence of thermal residual stress are analyzed. The computed results show that the presence of thermal residual stress induces asymmetric behavior in tension and compression. This depends intimately on the micro-structural parameters of the composite. Finite element calculations are also performed to predict the secant thermal dilatation coefficient and stress-strain relations in tension and compression for unidirectional composites. The proposed analytical method is found to compare favorably with the finite element results.

1 Introduction

Because of the difference in thermal dilatation coefficients, thermal residual stress can be generated during fabrication processes of composites. This thermal residual stress can alter the mechanical properties of the composites: asymmetric behavior in tension and compression, plastic flow near the interface between inclusions and matrix. There is a large body of works devoted both theoretically and experimentally to determine this residual stress and to analyze their consequences [1]–[5]. However most of the works are focussed on aligned composites, and few on the composite with fiber's orientation. Recently, Hu and Weng [6] showed that for a composite with two- or three-dimensional randomly oriented inclusions Mori-Tanaka's method [7] can not be realized from the microstructure proposed by Ponte Castaneda and Willis [8], thus it is desirable to use this new microstructure to consider the effect of fiber orientations on the effect of thermal stresses. In this paper, based on the method proposed by Hu and Weng [5] and the elastic model given by Ponte Castaneda and Willis [8], a method will be proposed to predict the influence of thermal residual stress on the elastic-plastic properties of composites. The emphasis will be placed on the effects of inclusion's orientation and distribution. In the end, the analytical results will be compared with some finite element calculations.

2 Theoretical formulation

2.1 Effective thermo-elastic properties of composites

Consider a representative volume element (RVE) of a composite, which consists of a continuous matrix and spheroidal inclusions. $\boldsymbol{L}_0, \boldsymbol{L}_r\ (r = 1, \ldots, N)$ denote the elastic moduli tensors of the matrix and the inclusion of type r, respectively, and c_r the volume concentration for the inclusion of type r. The corresponding stress-temperature tensors are denoted by $\underline{\underline{l}}_0, \underline{\underline{l}}_r$ $(r = 1, \ldots, N)$. Under a uniform temperature change θ, the constitutive relations for each phases are written by $\underline{\underline{\sigma}}_r = \boldsymbol{L}_r : \underline{\underline{\varepsilon}}_r - \underline{\underline{l}}_r \theta$, and we have $\underline{\underline{l}}_r = \boldsymbol{L}_r : \underline{\underline{\alpha}}_r$, where $\underline{\underline{\alpha}}_r$ is the thermal dilatation coefficient tensor for phase $r (r = 0, 1, \ldots, N)$. In this paper, the bold letter denotes a fourth-order tensor, $\underline{\underline{d}}$ denotes a second-order tensor, and $\underline{d}$ means a vector.

For the RVE under a macroscopic strain $\underline{\underline{E}}$ and a uniform temperature change θ, the local constitutive relation can be written as (the matrix is taken as the comparison material):

$$\underline{\underline{\sigma}} = \boldsymbol{L}_0 : \underline{\underline{\varepsilon}} - \underline{\underline{l}}_0 \theta + \underline{\underline{\tau}}, \tag{1}$$

where τ is the stress polarization, which, according to Willis [9], has the governing equation

$$(\boldsymbol{L} - \boldsymbol{L}_0)^{-1} : \underline{\underline{\tau}} + \int_{RVE} \Gamma^0(\underline{x}, \underline{x}') : \underline{\underline{\tau}}(\underline{x}')\, d\underline{x}' = \underline{\underline{E}} - (\boldsymbol{L} - \boldsymbol{L}_0)^{-1} : (\underline{\underline{l}} - \underline{\underline{l}}_0)\theta\,. \tag{2}$$

$\Gamma^0(\underline{x}, \underline{x}')$ is here the second derivative of Green's tensor function associated with the infinite medium $\boldsymbol{L}_0$.

Now following the micro-mechanical method proposed by Ponte Castaneda and Willis [8], and averaging Eq. (2) over the inclusion of type r and introducing the ellipsoidal distribution for the inclusions, we get

$$[(\boldsymbol{L}_r - \boldsymbol{L}_0)^{-1} + \boldsymbol{P}_i{}^r] : \underline{\underline{\tau}}^r - \sum_{s=1}^{N} c_s \boldsymbol{P}_d{}^{rs} : \underline{\underline{\tau}}^s = \underline{\underline{E}} - (\boldsymbol{L}_r - \boldsymbol{L}_0)^{-1} : (\underline{\underline{l}}_r - \underline{\underline{l}}_0)\,\theta\,, \tag{3}$$

where $\boldsymbol{P}_i{}^r = \int_{\Omega^r} \Gamma^0(\underline{x}, \underline{x}')\, d\underline{x}'$, $\underline{x} \in \Omega^r$ and $\boldsymbol{P}_d{}^{rs} = \int_{\Omega_d{}^{rs}} \Gamma^0(\underline{x}, \underline{x}')\, d\underline{x}'$, $\underline{x} \in \Omega_d{}^{rs}$. Ω^r is the region occupied by the inclusion of type r. If the conditional probability density is expressed by $p^{r|s}(\underline{z}'') = \varphi_{rs}(\underline{\underline{Z}}^{rs} : \underline{z}'')$, $p^{r|s}(\underline{z}'')$ is the conditional probability density for finding an inclusion of the type r centered at $\underline{z}$, provided that there is an inclusion of the type s centered at $\underline{z}'$, and $\underline{z}'' = \underline{z} - \underline{z}'$. The spheroids $\Omega_d{}^{rs}$, which characterize the distribution of the inclusions, are defined by $\Omega_d{}^{rs} = \{\underline{x} : |\underline{\underline{Z}}^{rs} : \underline{x}|^2 < 1\}$, and inside $\Omega_d{}^{rs}$, $p^{r|s}(\underline{z}'') = 0$.

To simplify the analysis, the spheroids $\Omega_d{}^{rs}$ are assumed to be the same for all inclusions, this leads to $\boldsymbol{P}_d{}^{rs} = \boldsymbol{P}_d$. From Eq. (3), one obtains:

$$\sum_{r=1}^{N} c_r \tau^r = \left[\boldsymbol{I} - \sum_{r=1}^{N} c_r \boldsymbol{T}^r \boldsymbol{P}_d\right]^{-1} \left[\sum_{r=1}^{N} c_r \boldsymbol{T}^r\right] : \underline{\underline{E}} - \left[\boldsymbol{I} - \sum_{r=1}^{N} c_r \boldsymbol{T}^r \boldsymbol{P}_d\right]^{-1} \times \left[\sum_{r=1}^{N} c_r \boldsymbol{T}^r (\boldsymbol{L}_r - \boldsymbol{L}_0)^{-1} : (\underline{\underline{l}}_r - \underline{\underline{l}}_0)\right] \theta,$$

where $\boldsymbol{T}^r = [(\boldsymbol{L}_r - \boldsymbol{L}_0)^{-1} + \boldsymbol{P}_i{}^r]^{-1}$.

So the effective thermo-elastic properties of the composite can be obtained by using the relation $\underline{\underline{\Sigma}} = \boldsymbol{L}_0 : \underline{\underline{E}} - \underline{\underline{l}}_0\theta + \sum_{r=1}^{N} c_r\tau^r$:

$$\boldsymbol{L}_{\mathrm{PCW}} = \boldsymbol{L}_0 + \left[\boldsymbol{I} - \sum_{r=1}^{N} c_r \boldsymbol{T}^r \boldsymbol{P}_d\right]^{-1} \left[\sum_{r=1}^{N} c_r \boldsymbol{T}^r\right], \tag{4}$$

$$\underline{\underline{l}}_{\mathrm{PCW}} = \underline{\underline{l}}_0 + \left[\boldsymbol{I} - \sum_{r=1}^{N} c_r \boldsymbol{T}^r \boldsymbol{P}_d\right]^{-1} \left[\sum_{r=1}^{N} c_r \boldsymbol{T}^r (\boldsymbol{L}_r - \boldsymbol{L}_0)^{-1} : (\underline{\underline{l}}_r - \underline{\underline{l}}_0)\right]. \tag{5}$$

In the following, the composite is assumed to consist of two isotropic phases. The moduli (compliance tensor) and the stress-temperature tensor of the matrix are denoted by $\boldsymbol{L}_0(\boldsymbol{M}_0)$ and $\underline{\underline{l}}_0$, and by $\boldsymbol{L}_1(\boldsymbol{M}_1)$ and $\underline{\underline{l}}_1$ for the inclusion. c_1 is the volume fraction of the inclusion. The inclusions can be aligned, two- or three-dimensional randomly oriented. In this case Eqs. (4), (5) can be rewritten as

$$\boldsymbol{L}_{\mathrm{PCW}} = \boldsymbol{L}_0 + c_1[\boldsymbol{I} - c_1 < \boldsymbol{T}^1 > \boldsymbol{P}_d]^{-1} < \boldsymbol{T}^1 >, \tag{6}$$

$$\underline{\underline{l}}_{\mathrm{PCW}} = \underline{\underline{l}}_0 + c_1[\boldsymbol{I} - c_1 < \boldsymbol{T}^1 > \boldsymbol{P}_d]^{-1} < \boldsymbol{T}^1(\boldsymbol{L}_1 - \boldsymbol{L}_0)^{-1} : (\underline{\underline{l}}_1 - \underline{\underline{l}}_0) > . \tag{7}$$

$< \bullet >$ means the orientational average. It is seen that for the aligned composite Eq. (7) is identical to that given by Laws [10], which has the following form:

$$\underline{\underline{l}}_{\mathrm{PCW}} = \underline{\underline{l}}_0 + (\boldsymbol{L}_{\mathrm{PCW}} - \boldsymbol{L}_0)(\boldsymbol{L}_1 - \boldsymbol{L}_0)^{-1} : (\underline{\underline{l}}_1 - \underline{\underline{l}}_0). \tag{8}$$

For the studied composite with isotropic phases, the operator of the orientational average works only on $\boldsymbol{T}^1$ in Eq. (7), so for the composite with aligned, two- or three-dimensional randomly oriented inclusions, Eq. (8) still holds.

The average stresses in the inclusions and the matrix can be obtained by using the constitutive equations for each phase. From definition

$$\underline{\underline{\Sigma}} = (1 - c_1)\underline{\underline{\sigma}}_0 + c_1 < \underline{\underline{\sigma}} >_1, \qquad \underline{\underline{E}} = (1 - c_1)\underline{\underline{\varepsilon}}_0 + c_1 < \underline{\underline{\varepsilon}} >_1 \tag{9}$$

and with the help of the equations $\underline{\underline{E}} = \boldsymbol{M}_{\mathrm{PCW}} : \underline{\underline{\Sigma}} + \underline{\underline{\alpha}}_{\mathrm{PCW}}\theta$, $\underline{\underline{\varepsilon}}_0 = \boldsymbol{M}_0 : \underline{\underline{\sigma}}_0 + \underline{\underline{\alpha}}_0\theta$ and $< \underline{\underline{\varepsilon}} >_1 = \boldsymbol{M}_1 : < \underline{\underline{\sigma}} >_1 + \underline{\underline{\alpha}}_1\theta$, the average stress in the inclusions can be calculated by

$$\begin{aligned} < \underline{\underline{\sigma}} >_1 &= \frac{1}{c_1}[\boldsymbol{M}_1 - \boldsymbol{M}_0]^{-1}[\boldsymbol{M}_{\mathrm{PCW}} - \boldsymbol{M}_0] : \underline{\underline{\Sigma}} + \frac{1}{c_1}[\boldsymbol{M}_1 - \boldsymbol{M}_0]^{-1} : [\underline{\underline{\alpha}}_{\mathrm{PCW}} - c_1\underline{\underline{\alpha}}_1 - (1 - c_1)\underline{\underline{\alpha}}_0]\theta \\ &= \boldsymbol{Q} : \underline{\underline{\Sigma}} + \underline{\underline{R}}\theta, \end{aligned} \tag{10}$$

where $\boldsymbol{M}_{\mathrm{PCW}} = \boldsymbol{L}_{\mathrm{PCW}}^{-1}$, $\underline{\underline{\alpha}}_{\mathrm{PCW}} = \boldsymbol{M}_{\mathrm{PCW}} : \underline{\underline{l}}_{\mathrm{PCW}}$.

The elastic moduli and the thermal dilatation coefficient of the composite can be computed with the aid of Eqs. (6), (8), in which the distribution of the inclusions is considered through tensor $\boldsymbol{P}_d$. According to [11], tensor $\boldsymbol{P}_d$ can be related to Eshelby tensors by $\boldsymbol{P}_i^1 = \boldsymbol{S}\boldsymbol{L}_0^{-1}$, $\boldsymbol{P}_d = \boldsymbol{S}_d\boldsymbol{L}_0^{-1}$, where $\boldsymbol{S}, \boldsymbol{S}_d$ are Eshelby tensors, which depend on the moduli of the matrix and the aspect ratios w, w_d of the spheroids Ω and Ω_d. For an isotropic matrix, the Eshelby tensor has a simple analytical form [12].

2.2 *Homogenized effective stress of ductile matrix*

In this paper, the method based on second-order moment of stress developed by Qiu and Weng [13], Suquet [14], and Hu [15] will be utilized to compute the average effective stress in the matrix. According to Hu and Weng [5], for a composite under a macroscopic loading $\underline{\underline{\Sigma}}$

and a uniform temperature change θ, the homogenized effective stress in the matrix can be calculated by:

$$\sigma_{\text{eff}}^2 = \underline{\underline{\Sigma}} : \left[-\frac{3\mu_0{}^2}{1-c_1} \frac{\partial \boldsymbol{M}_{\text{PCW}}}{\partial \mu_0} \right] : \underline{\underline{\Sigma}} + \theta(\underline{\underline{\alpha}}_1 - \underline{\underline{\alpha}}_0) : \left[-\frac{6c_1\mu_0{}^2}{1-c_1} \frac{\partial \boldsymbol{Q}}{\partial \mu_0} \right] : \underline{\underline{\Sigma}} + (\underline{\underline{\alpha}}_1 - \underline{\underline{\alpha}}_0) : \left[-\frac{3c_1\mu_o{}^2}{1-c_1} \frac{\partial \underline{\underline{R}}}{\partial \mu_0} \right] \theta^2 ,$$

where $\mu_0, \varkappa_0$ are respectively the shear and bulk moduli of the matrix. By setting the above effective stress equal to the initial yield stress of the matrix ($\sigma_{\text{eff}} = \sigma_y$), the macroscopic yield function of the composite can be determined, and the influence of the thermal residual stress and inclusion's orientation and distribution can be analyzed.

When a plastic deformation takes place in the matrix, its secant moduli will be utilized to consider the change of the constraint on the inclusions. Assuming the power law type hardening for the matrix $\sigma_e = \sigma_y + h\varepsilon_e{}^n$, where σ_e, ε_e are von Mises effective stress and effective plastic strain, the secant shear and bulk moduli of the matrix $\mu_0{}^s, \varkappa_0{}^s$ at ε_e are of the following forms (the plastic incompressibility is assumed for the matrix):

$$\mu_0{}^s = \frac{1}{\left(\frac{1}{\mu_0} + \frac{3\varepsilon_e}{(\sigma_y + h(\varepsilon_e)^n)} \right)} , \qquad \varkappa_0{}^s = \varkappa_0 . \tag{11}$$

Using the secant moduli tensor $\boldsymbol{L}_0{}^s$ instead of $\boldsymbol{L}_0$, the plastic deformation of the composite can be decomposed into a series of elastic problems (for more details, see [16], [17]). In the presence of thermal-residual stress, we follow the method developed by Bhattacharyya et al. [18], and Hu and Weng [5].

3 Numerical applications

The proposed analytical model will be applied to a SiC/Al composite system. The elastic constants for the inclusions are: $E_1 = 490$ GPa, $\nu_1 = 0.17$; and the constants for the matrix: $E_0 = 68.3$ GPa, $\nu_0 = 0.33$; $\sigma_y = 250$ MPa, $h = 577$ MPa, $n = 0.355$, and the thermal coefficients are $\alpha_0 = 21.6 \times 10^{-6}/°\text{C}$, $\alpha_1 = 3.8 \times 10^{-6}/°\text{C}$. The volume fraction is kept constant $c_1 = 15\%$ and $\theta = -500$ °C in the following analysis. The method for computing the orientational average can be found in [6].

Figure 1 shows the secant thermal dilatation coefficients $\alpha_e{}^L$ (fiber direction for aligned composites, in-plane for composites with 2-d randomly oriented inclusions and overall for the 3-d orientations). Here the inclusion's distribution is assumed as isotropic, $w_d = 1$. For prolate inclusions ($w > 1$), the composites with aligned inclusions gives minimum $\alpha_e{}^L$, and those with three-dimensional randomly oriented inclusions predict maximum $\alpha_e{}^L$. The results are reversed for the composites with oblate inclusions.

Figure 2 illustrates stress-strain curves in tension and compression for the composites with aligned, 2- or 3-d randomly oriented inclusions ($w_d = 1, w = 2$). Here the white circles show the distribution of inclusions (w_d), whereas the dark regions represent the shape of the inclusions (w). For the composites with aligned and 2-d randomly oriented inclusions, owing to the presence of thermal residual stress, the compressive curves are superior to the tension ones, and for the composite with 3-d randomly oriented inclusions, the difference in tension and compression is small, but the tension curve is a little superior to the compressive one. The

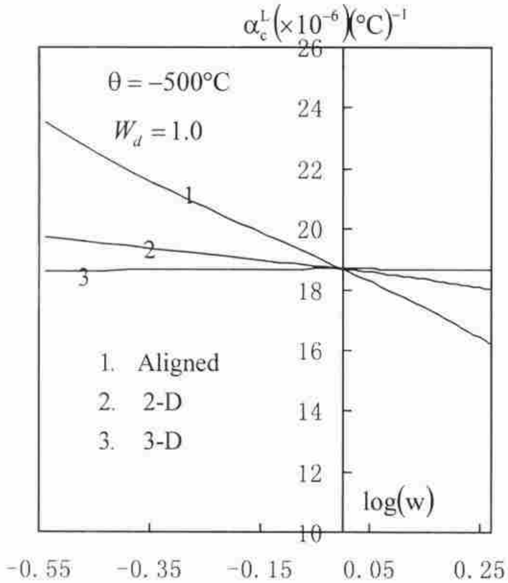

Fig. 1. Secant dilatation coefficient of the composite as function of the aspect ratio of inclusions ($\theta = -500$ °C, $w_d = 1$)

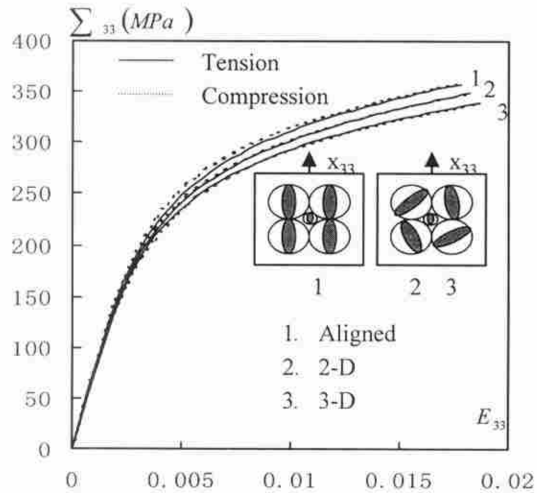

Fig. 2. Stress-strain curves in tension and compression of the composites for different inclusion's orientations ($\theta = -500$ °C, $w_d = 1, w = 2$)

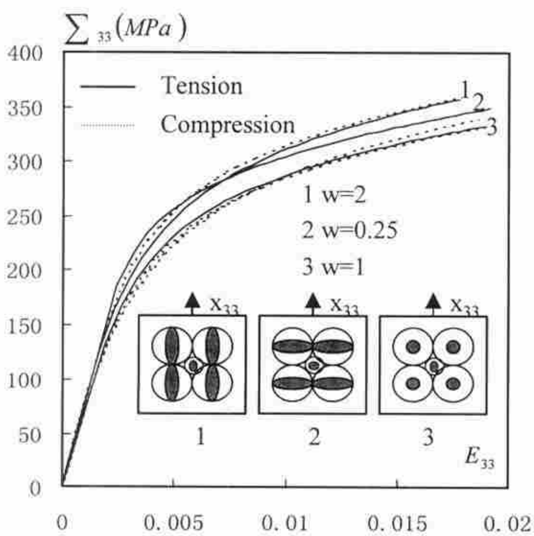

Fig. 3. Stress-strain curves in tension and compression of the composites with aligned inclusions of different aspect ratios ($w = 0.25, w = 1.0, w = 2$ and $w_d = 1$, $\theta = -500$ °C)

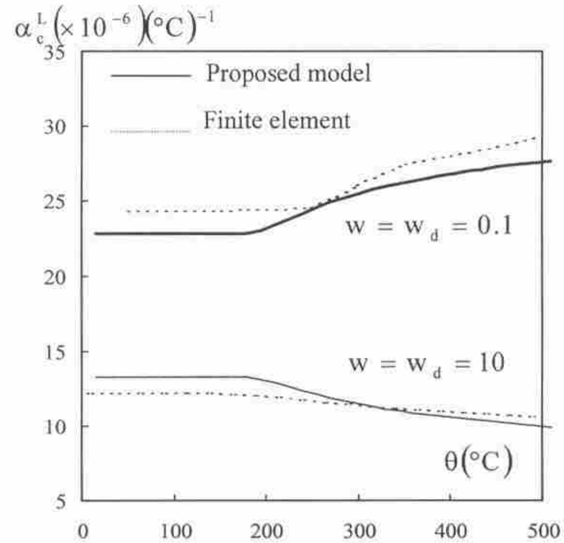

Fig. 4. Comparison of secant dilatation coefficients for aligned composites between the proposed model and finite element model

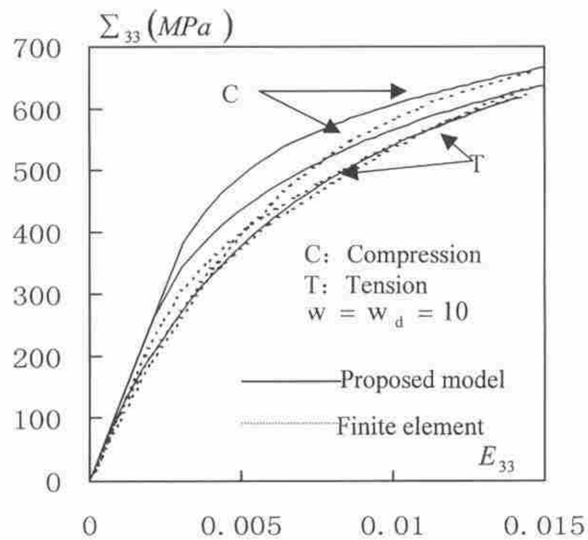

Fig. 5. Comparison of stress-strain curves in tension and compression with and without residual stress for aligned composite ($w = 10$) between the proposed method and finite element model

stress-strain curves in tension and compression for a composite with aligned inclusions for the inclusion's aspect ratios $w = 2, w = 1, w = 0.25$ are depicted in Fig. 3 ($w_d = 1$). It is seen that the difference is most pronounced for $w = 0.25$, and that the tension curve in this case is superior to the compressive one.

It must be mentioned that the elastic model proposed by Ponte Castaneda and Willis assumed the ellipsoidal symmetry for the distribution of the inclusions, and that this limits significantly the possible choice of the inclusion's aspect ratio for a fixed distribution of the inclusions [8].

Finally the proposed analytical method is also compared with a finite element calculation (unit cell model) for the composites with aligned inclusions. The spheroid Ω_d characterizing the distribution of inclusions is taken to have the same form as the inclusion Ω; this leads to $w_d = w$. In this case, the proposed model corresponds to the Mori-Tanaka mean field theory

[6], [8], [11]. The finite element calculation is carried out by ANSYS finite element program. Figure 4 gives the comparison results for the secant dilatation coefficient ${\alpha_c}^L$ for the composites with $w_d = w = 10, 0.1$. For the composite with the aligned oblate inclusions ($w = 0.1$), ${\alpha_c}^L$ is larger than that of the matrix. The plastic deformation of the matrix increases ${\alpha_c}^L$. The situation is reversed for the composite with aligned prolate inclusions ($w = 10$). The comparison results of the stress-strain curves in tension and compression for the composite with and without thermal residual stress are also given in Fig. 5 ($w = 10$); it is seen that the predictions by the proposed analytical model agree quantitatively with those given by the finite element model.

4 Conclusions

An analytical method has been proposed to predict the influence of residual stress on the elastic-plastic behavior of a general composite. The model incorporates the micro-structural parameters like fiber shape, orientation, distribution and volume fraction. The computed results show that the influence of residual stress on the macroscopic properties depends closely on the microstructures of the composite. The predicted results of secant dilatation coefficient and stress-strain curves for aligned composites agree reasonably with those calculated by the finite element method.

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