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Size-dependence of overall in-plane plasticity for fiber composites

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Abstract

For composite materials, a size-dependent effective behavior may manifest when the particle size has the same order as the intrinsic length scale (grain size for a polycrystalline material for example) of the matrix material. This size-dependent effective nonlinear property of a fiber composite is investigated by an analytical micromechanical method. The non-local effect of the matrix with a coarse-grain microstructure is considered by idealizing it as a micropolar material. Mori–Tanaka's method and generalized self-consistent method are extended to a micropolar fiber composite, the effective shear and in-plane bulk moduli are obtained analytically. The results show that the effective in-plane shear modulus is large for the composites with small diameter fibers, and the effective in-plane bulk modulus will not depend on the fiber size. We further extend the secant moduli method based on second-order stress moment to micropolar composites. Size-dependent yield functions and effective stress and strain relations of a micropolar fiber composite are derived in an analytical way. The size dependence is more pronounced for the composite reinforced by hard fibers and for shear loading. The proposed method shares the same structure as in the classical micromechanics, and when the fiber size is very large compared to the intrinsic length of the matrix, the classical micromechanics method can be recovered, as expected.

Keywords: Micromechanics; Fiber composite; Plasticity; Micropolar; Size effect

1. Introduction

Prediction of overall properties for composite materials has seen a rapid development during the last decade. The systematic methods developed, usually termed as micromechanics, consist of different strategies and techniques to bridge the macroscopic (overall) properties from the information of local constituents and microstructures. These methods can be roughly classified into the following four groups: (a) Universal exact relations independent of microstructures, which are initiated from the CLM theorem

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(Cherkaev et al., 1992). Some interesting results are, for example, the effective in-plane Young's modulus (E_c) is independent of the matrix Poisson's ratio for a planar isotropic voided (cracked) material with an isotropic matrix (Cherkaev et al., 1992), and the effective Poisson's ratio (ν_c) must be a linear function of the matrix Poisson's ratio (ν_0) with a proportional coefficient E_c/E_0 regardless of the shape and the concentration of voids (cracks) (Hu and Weng, 2001). The detailed results devoted to this aspect can be found in the recent monograph given by Milton (2002). (b) Bounding methods, whose objective is to give the range of the effective modulus if only partial microstructural information is available. These include, for example, the Hashin-Shtrikman bounds for the isotropic distribution of phases, and the third-order bounds incorporating the three-point correlation functions of microstructures, as discussed extensively by Torquato (2002) and Nemat-Nasser and Hori (1993). (c) Approximate methods, whose idea is to simplify the complex interaction between phases by some typical morphologies (or patterns), and a single such pattern is then put into a reference material to determine the localization relation. These include Mori–Tanaka's method, self-consistent method, generalized self-consistent and double inclusion methods, which bear certain connections as established by Hu and Weng (2000). These approximate methods are discussed in detail by Nemat-Nasser and Hori (1993). (d) Computational methods, which make use of numerical techniques to evaluate the localization relation for a more realistic microstructure, as recently summarized by Schmauder (2002). All these methods provide powerful tools for the design of composite materials by tailoring their microstructure.

The above-mentioned methods share the following assumption: the local constituents and the homogenized material can be idealized as Cauchy materials without any inner microstructure. This homogenization method is adequate for the case where the structure length scale is much larger than the intrinsic length scales of the material micromorphology (Eringen, 1999). For a composite of an inclusion-matrix morphology, whose particle or fiber size is much larger than the intrinsic length of the matrix (for example, grain size for a polycrystalline material), the effective response of the matrix material itself can be described by using a Cauchy material model. As shown in Fig. 1, if we denote L as the structural length scale, l as the size of a representative volume element (RVE), A as the size of the reinforced phase, l_m as the intrinsic length scale of the matrix, the classical micromechanics applies for the length scale condition $L \gg l \gg A \gg l_m$. In this case, both the matrix material and the homogenized macro-element can be idealized as Cauchy materials without any inner microstructure. However if the size of the reinforced phase is comparable to the intrinsic length scale of the matrix $A \approx l_m$, a size-dependent overall behavior is usually observed for these composites. This size dependence is well-known for polycrystalline materials (Hall-Petch relation), and metal matrix composites. The composite with small particles has a high flow stress compared to that with large particles at the same volume concentration (Kouzeli and Mortensen, 2002). In this situation, as shown in Fig. 1, the non-local nature of the matrix material due to its coarse-grain microstructure should be included in a proper theoretical formulation. This problem is also relevant to nanocomposite materials, biological materials (Sharma and Dasgupta, 2002).

In order to explain the enhanced flow stress of the composite due to the decreasing particle size, two strategies are usually adopted: one is from a material science point of view, which considers that, in

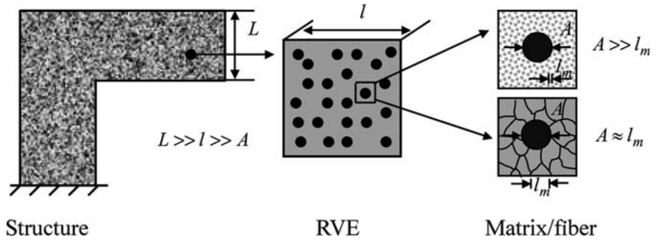

Fig. 1. Length scale relations.

additional to the storage dislocation density, the geometrically necessary dislocation density is believed to contribute the hardening of a material (Fleck et al., 1994); the other is from a continuum mechanics point of view, which argues that the non-local aspect of the matrix material should be included in a continuum formulation, such that high-order continuum models must be proposed to describe the response of the matrix material (Smyshlyaev and Fleck, 1995; Shu and Barlow, 2000; Forest et al., 2000; Wei, 2001; Chen and Wang, 2002; Liu and Hu, in press). There are different high-order theories presented in literature, namely: non-local theory, gradient plasticity, strain gradient theory, and micropolar theory. In the first theory (Bazant and Pijaudier-Cabot, 1988), the response of a material at a point is determined not only by the state at that point but also by the deformation of its neighborhood. The second theory proposed by Aifantis (1984) includes the gradients of state variables to consider the non-local behavior of a material. The third one (Fleck and Hutchinson, 1993a,b; Gao et al., 1999) introduces higher-order strain gradients in addition to the original strain and stress measures. In micropolar theory (Eringen, 1999), three rotational degrees of freedom are introduced in addition to the conventional three displacements at each material point, and this leads to a non-symmetric stress (strain) and a high-order couple stress (torsion).

The micropolar theory will be undertaken in this paper to consider the non-local response of the matrix material. Also concerned with the overall property for a micropolar composite, Yuan and Tomita (2001) utilized a unit-cell model to evaluate the effective elastic property of a micropolar matrix with periodic voids. Recently Sharma and Dasgupta (2002), Liu and Hu (in press) independently proposed to extend Mori–Tanaka's method to compute the effective property of a particulate micropolar composite. The prediction on the overall elastic property of a fiber micropolar composite were also discussed by Sharma and Dasgupta (2002) and Xun et al. (2004) respectively. To predict the nonlinear behavior of micropolar composites, Chen and Wang (2002) used a finite element method and unit cell model to analyze the size-dependent overall nonlinear behavior of a short fiber composite. Recently Liu and Hu (in press) proposed an analytical micromechanical model by extending the classical secant moduli method based on second-order stress moment (Qiu and Weng, 1992; Suquet, 1995; Hu, 1996) to a micropolar composite, and by this method the influence of the particle size on the nonlinear overall behavior of the composite can be captured in an analytical form.

In this paper, we will consider the following length scale condition $L \gg l \gg A \approx l_{\mathrm{m}}$. The matrix material will be considered as a nonlinear micropolar material, and the overall property of the composite can still be idealized as a Cauchy continuum due to the small size of the RVE compared to that of the structure. The method proposed by Liu and Hu (in press) will be extended in this paper for a fiber reinforced composite. A generalized self-consistent method for micropolar composites will also be proposed, and an analytical expression for the size-dependent yield function of the fiber composite will be derived. The size dependence of the nonlinear overall behavior of the composite will be analyzed by the secant-moduli method based on second-order stress and couple stress moments.

2. Micropolar elasticity and plasticity

We will in the following briefly recall some essential elements of micropolar theory in a plane-strain condition, but for more details, the readers are encouraged to refer to the monographs by Eringen (1999) and Nowacki (1986). It is assumed in micropolar theory that the inner microstructure inside a material point can have an independent rigid rotation; therefore there are not only forces but also moments that can be transmitted across a surface of a material element. A well-posed two-dimensional micropolar boundary-value problem is described by the following three sets of governing equations (body force and couple are neglected):

$$\varepsilon_{\alpha\beta} = u_{\alpha,\beta} - e_{3\alpha\beta}\phi_3, \quad k_{\alpha 3} = \phi_{3,\alpha} \qquad \textit{Kinematical relations} \tag{1}$$

$$\sigma_{\alpha\beta,\beta} = 0, \quad m_{\beta 3,\beta} + e_{3\alpha\beta}\sigma_{\alpha\beta} = 0 \qquad \textit{Equilibrium conditions} \tag{2}$$

Constitutive relations

$$\sigma_{\alpha\beta} = \lambda\varepsilon_{\zeta\zeta}\delta_{\alpha\beta} + (\mu+\kappa)\varepsilon_{\alpha\beta} + (\mu-\kappa)\varepsilon_{\beta\alpha} \tag{3a}$$

$$m_{\alpha 3} = 2\beta\phi_{3,\alpha} \tag{3b}$$

$$m_{3\alpha} = 2\gamma\phi_{3,\alpha} \tag{3c}$$

With the following boundary conditions:

$$\sigma_{\alpha\beta}n_\beta = p_\alpha, \quad m_{\alpha 3}n_\alpha = z \qquad \text{on } \Gamma^\sigma \tag{4}$$

$$u_\alpha = u_\alpha^b, \quad \phi_3 = \phi_3^b \qquad \text{on } \Gamma^u \tag{5}$$

where u_α, ϕ_3 are respectively macroscopic displacements of a material point and a microscopic rotation angle for the microstructure inside of this material point. $\varepsilon_{\alpha\beta}$, $k_{\alpha 3}$ are the strain and the torsion, their thermodynamically conjugated variables are respectively the stress $\sigma_{\alpha\beta}$ and the couple stress $m_{\alpha 3}$, which are usually asymmetric. p_β and z are the surface forces and the moment vector, and n_α is an exterior unit normal. $e_{3\alpha\beta}$ is the third-order permutation tensor. μ, λ are the classical Lame's constants, while κ, γ, β are the new material constants introduced in micropolar theory. $\delta_{\alpha\beta}$ is the two-dimensional Kroneker Delta. Greek letter indices are from 1 to 2.

If we denote $s_{(\alpha\beta)}$, $\sigma_{\langle\alpha\beta\rangle}$, σ $(\sigma \equiv 1/2\sigma_{\beta\beta})$ and $e_{(\alpha\beta)}$, $\varepsilon_{\langle\alpha\beta\rangle}$, ε $(\varepsilon \equiv 1/2\varepsilon_{\beta\beta})$ respectively the deviatoric of the symmetric, the anti-symmetric and the hydrostatic parts of the stress and strain tensors, the isotropic constitutive equations can be rewritten in the following form:

$$s_{(\alpha\beta)} = 2\mu e_{(\alpha\beta)}, \quad \sigma_{\langle\alpha\beta\rangle} = 2\kappa\varepsilon_{\langle\alpha\beta\rangle}, \quad \sigma = 2k\varepsilon$$

$$m_{\alpha 3} = 2\beta k_{\alpha 3}, \quad m_{3\alpha} = 2\gamma k_{\alpha 3} \tag{6}$$

where $k = \lambda + \mu$ is the in-plane bulk modulus.

The elastic strain potential for a micropolar material of the first kind in a plane-strain condition is written as (Nowacki, 1986)

$$w = \frac{1}{4\mu}s_{(\alpha\beta)}s_{(\alpha\beta)} + \frac{1}{4\kappa}\sigma_{\langle\alpha\beta\rangle}\sigma_{\langle\alpha\beta\rangle} + \frac{1}{2k}\sigma^2 + \frac{1}{4\beta}m_{\alpha 3}m_{\alpha 3} \tag{7}$$

Because of the dimension mismatch for the two sets of moduli, some elastic intrinsic lengths of a micropolar material can be defined. They can be defined in different ways, and in this paper, they are simply given as

$$l_1^2 = \frac{\beta}{\mu}, \quad l_2^2 = \frac{\gamma}{\mu} \tag{8}$$

In the case of plasticity, following Fleck and Hutchinson (1993a,b), and Liu and Hu (in press), we will extend the classical Von Mises criterion to a micropolar material by defining an equivalent stress

$$\tilde{\sigma}_e = \sqrt{\frac{3}{2}\left[\left(s_{(\alpha\beta)}s_{(\alpha\beta)} + s_{33}s_{33}\right) + \frac{1}{l_p^2}m_{\alpha 3}m_{\alpha 3}\right]} \tag{9}$$

where l_p is introduced for dimensional consistency and can be considered as a plastic length scale. In the following, for simplification, only elastically incompressible matrix is considered, in this case the equivalent stress becomes:

$$\tilde{\sigma}_e = \sqrt{\frac{3}{2}\left[s_{(\alpha\beta)}s_{(\alpha\beta)} + \frac{1}{l_p^2}m_{\alpha3}m_{\alpha3}\right]} \tag{10}$$

With the help of the above-defined equivalent stress, the nonlinear potential energy for a micropolar material can be written as

$$w = w_0(\tilde{\sigma}_e) + \frac{1}{2k}\sigma^2 + \frac{1}{4\kappa}\sigma_{\langle\alpha\beta\rangle}\sigma_{\langle\alpha\beta\rangle} \tag{11}$$

For a power-type hardening law, $w_0(\tilde{\sigma}_e)$ is given by

$$w_0(\tilde{\sigma}_e) = \frac{\tilde{\sigma}_e^2}{6\mu} + \frac{n}{n+1}\frac{1}{H^{1/n}}(\tilde{\sigma}_e - \sigma_y)^{\frac{n+1}{n}} \tag{12}$$

where σ_y, H, n are respectively yield stress and hardening parameters determined from a uniaxial tensile test.

Nonlinear stress and strain relations can be simply obtained by differentiating the potential energy, and this gives

$$e_{(\alpha\beta)} = \frac{\partial w}{\partial s_{(\alpha\beta)}} = \frac{3}{2\tilde{\sigma}_e}\frac{\partial w_0}{\partial\tilde{\sigma}_e}s_{(\alpha\beta)}, \quad \varepsilon_{\langle\alpha\beta\rangle} = \frac{1}{2\kappa}\sigma_{\langle\alpha\beta\rangle}, \quad k_{\alpha3} = \frac{3}{2l_p^2\tilde{\sigma}_e}\frac{\partial w_0}{\partial\tilde{\sigma}_e}m_{\alpha3}, \quad \varepsilon = \sigma/2k \tag{13}$$

From the deformation theory of micropolar plasticity given by Eq. (13), the secant moduli of the nonlinear micropolar material can be defined as

$$\mu^s = \frac{1}{(1/\mu) + 3[(\tilde{\sigma}_e - \sigma_y)/H]^{1/n}/\tilde{\sigma}_e}, \quad \kappa^s = \kappa, \quad \beta^s = l_p^2\mu^s, \quad k^s = k \tag{14}$$

The superscript 's' means the corresponding secant quantity, and from the argument that the constant β remains continuous from elasticity to plasticity, we have $l_p = l_1$. For further simplification, we assume that $l_p = l_1 = l_2 = l_m$, so only one length scale appears in the proposed theory. These secant moduli will be used to estimate the overall elasto-plastic behavior of micropolar fiber composites.

3. Elastic moduli of a micropolar fiber composite

3.1. Mori–Tanaka's estimate and generalized self-consistent estimate

As discussed in the introduction, the following length scale condition $L \gg l \gg R \approx l_m$ is considered. In this case, the RVE of the composite material is small enough, and the classical symmetric uniform stress (or strain) condition is applied on its boundary. The composite as a whole can be considered as a classical Cauchy continuum (Forest et al., 1999; Liu and Hu, in press), characterized by the effective in-plane bulk modulus and shear modulus k_c, μ_c respectively. However since the fiber diameter is comparable to the intrinsic length scale of the surrounding matrix (see Fig. 1 $R \approx l_m$), the non-local effect of the matrix may become important, and this non-local effect of the matrix material will be considered by micropolar theory in this paper.

The effective in-plane bulk and shear moduli k_c, μ_c of a micropolar composite can be evaluated by a proper localization relation and a homogenization technique, which are discussed by Sharma and Dasgupta (2002), Liu and Hu (in press), and Xun et al. (2004). The main idea can be summarized as follows: for a RVE of the micropolar composite under a uniform symmetric stress or strain boundary condition $\bar{\sigma}_{(\alpha\beta)}(\bar{\varepsilon}_{(\alpha\beta)})$, and the couple stress (or micro-rotation angle) on the boundary of the RVE is zero (since the

RVE is small, the stress on its boundary can be considered to be uniform). It can be shown that $\langle\sigma_{(\alpha\beta)}\rangle = \bar{\sigma}_{(\alpha\beta)}(\langle\varepsilon_{(\alpha\beta)}\rangle = \bar{\varepsilon}_{(\alpha\beta)})$, where $\langle\bullet\rangle$ means the volume average of the said quantity over the RVE. If the local stress $\sigma_{\alpha\beta}$ (non-symmetric) under the above prescribed boundary condition are known, their averages over the different phases can then be written as for an isotropic composite $\langle s_{(\alpha\beta)}\rangle_i = H_i^s\bar{s}_{(\alpha\beta)}$, $\langle\sigma\rangle_i = H_i^\sigma\bar{\sigma}$, which are called localization relations, where $\langle\bullet\rangle_i$ means the volume average of the said quantity over the phase i, $\bar{s}_{(\alpha\beta)}$ and $\bar{\sigma}$ are the deviatoric and the hydrostatic parts of the macroscopic stress $\bar{\sigma}_{(\alpha\beta)}$. With the help of the localization relation, the effective moduli of the micropolar composite can be evaluated by the same method as in the classical micromechanics, namely, $\bar{e}_{(\alpha\beta)} = \langle e_{(\alpha\beta)}\rangle = (\sum_i f_i \frac{1}{2\mu_i} H_i^s)\bar{s}_{(\alpha\beta)}$ for the effective shear modulus, and $\bar{\varepsilon} = \langle\varepsilon\rangle = (\sum_i f_i \frac{1}{2k_i} H_i^\sigma)\bar{\sigma}$ for the effective in-plane bulk modulus, where f_i μ_i, k_i are respectively the volume concentration, shear modulus and in-plane bulk modulus of the phase i. As in classical micromechanics, to determine the effective in-plane moduli of the micropolar composite, the key point is to evaluate the stress concentration factors H_i^s, H_i^σ. To this end, there are two methods for calculating the average stress in a fiber embedded in an infinite micropolar material: the first one is the average equivalent inclusion method based on the Eshelby tensor derived by Cheng and He (1997) for a micropolar material, and the second method is to directly determine the local stress and couple stress in a fiber by the potential functions proposed by Eringen (1999), and then to average them over the fiber region. The detail solution for the first method is given by Sharma and Dasgupta (2002) and Xun et al. (2004) respectively, and for the latter it is also presented by Xun et al. (2004). We recall only some final results, for a coated fiber embedded in an infinite material (the three phases can be micropolar materials) under a remote uniform symmetric stress $\bar{\sigma}_{(\alpha\beta)} = \bar{s}_{(\alpha\beta)} + \bar{\sigma}\delta_{\alpha\beta}$ and zero couple stress. With the help of the potential functions proposed by Eringen (1999) and the continuity conditions at the interfaces, we can determine the stress and couple stress in each phases, then the average stresses in the fiber and in the coated layer can be written in the following form:

$$\langle s_{(\alpha\beta)}\rangle_1 = p_1\bar{s}_{(\alpha\beta)}, \quad \langle\sigma\rangle_1 = s_1\bar{\sigma}; \qquad \langle s_{(\alpha\beta)}\rangle_2 = p_2\bar{s}_{(\alpha\beta)}, \quad \langle\sigma\rangle_2 = s_2\bar{\sigma} \tag{15}$$

where p_1, s_1, p_2, s_2 are the stress concentration coefficients, depending on the modulus tensors of each phase and on the relative portion of the fiber to the encircled matrix (it is set to be the volume fraction of the fiber in the composite), their evaluation is given in detail by Xun et al. (2004) for a micropolar composite, and shortly summarized in Appendix A, where indices 1, 2 denote the fiber, the matrix respectively.

For one single fiber embedded in a micropolar matrix, the average stresses in the fiber evaluated by the average equivalent inclusion method are (Sharma and Dasgupta, 2002; Xun et al., 2004)

$$\langle s_{(\alpha\beta)}\rangle_1 = \frac{1}{\mu_2/\mu_1 + (1-\mu_2/\mu_1)\langle K_{1212}^s\rangle_1}\bar{s}_{(\alpha\beta)} \tag{16a}$$

$$\langle\sigma\rangle_1 = \frac{1}{k_2/k_1 + (1-k_2/k_1)\langle K_{\alpha\alpha\beta\beta}^s\rangle_1}\bar{\sigma} \tag{16b}$$

where

$$\langle K_{1212}^s\rangle_1 = \frac{k_2+2\mu_2}{4(\mu_2+k_2)} - \frac{\kappa_2}{2(\mu_2+\kappa_2)}I_1(\eta)K_1(\eta), \quad \langle K_{\alpha\alpha\beta\beta}^s\rangle_1 = \frac{2k_2}{k_2+\mu_2}$$

They are the components of the average Eshelby tensor for a micropolar material with a cylindrical inclusion, where $I_M(\eta)$ is the modified Bessel function of the first kind of order M, $K_M(\eta)$ is the modified Bessel function of the second kind of order M, $\eta = a/g$, a means fiber radius, $g^2 = (\mu_2+\kappa_2)\beta_2/2\mu_2\kappa_2$.

With these localization relations, we are ready to derive the effective in-plane moduli of a micropolar composite. In the following, Mori–Tanaka's method (Mori and Tanaka, 1973) and generalized self-consistent method (Christensen and Lo, 1979) will be extended to evaluate the effective moduli of a micropolar fiber composite. The volume fraction of the fiber is f_1 and that of the matrix is denoted by $f_2(f_1+f_2=1)$.

Following the same procedure as in the classical micromechanics, and with the help of the solutions for a single fiber problem, we can obtain the effective moduli of a micropolar fiber composite.

3.1.1. Mori–Tanaka's method (average equivalent inclusion method)

Using the localization relation given by (16), the effective shear and in-plane bulk moduli of a micropolar fiber composite estimated by Mori–Tanaka's method are respectively:

$$\mu_c = \mu_2\left\{1 + \frac{f}{2(1-f)\langle K^s_{1212}\rangle_1 + [\mu_2/(\mu_1-\mu_2)]}\right\}, \tag{17a}$$

$$k_c = k_2\left\{1 + \frac{f}{[(1-f)\langle K^s_{\alpha\alpha\beta\beta}\rangle_1/2] + [k_2/(k_1-k_2)]}\right\}. \tag{17b}$$

3.1.2. Generalized self-consistent method

Following the idea of the classical micromechanics, we take the infinite material to be yet unknown composite of the moduli μ_c, k_c. As shown by Herve and Zaoui (1990), the energy approach and the average stress–strain approach are equivalent for the classical generalized self-consistent method. In this paper, the average stress–strain approach will be employed, and this means that the average stress and strain over the fiber and the encircled matrix (coated fiber pattern) are related by the yet-unknown composite moduli. This leads to the estimation of the effective in-plane bulk and shear moduli for a two-phase micropolar composite as

$$k_c = \frac{f_1 s_1 + f_2 s_2}{f_1\frac{s_1}{k_1} + f_2\frac{s_2}{k_2}}, \quad \mu_c = \frac{f_1 p_1 + f_2 p_2}{f_1\frac{p_1}{\mu_1} + f_2\frac{p_2}{\mu_2}} \tag{18}$$

Since the stress concentration coefficients p_1, s_1, p_2, s_2 are also functions of μ_c, k_c, so Eq. (18) provides in fact two equations to determine the unknown effective moduli μ_c, k_c. It is found that the effective in-plane bulk modulus predicted by both Mori–Tanaka's method and the generalized self-consistent method is the same as that predicted by the corresponding classical micromechanical methods. This result can be expected, since in micropolar theory the motion of the microstructure inside of a material point is taken to be a rigid rotation, the dilatational effect of the microstructure is neglected. In the following, only the effective shear modulus and the macroscopic shear stress and strain relation are examined.

3.2. Numerical application

In what follows, we will compare the predictions by the proposed two methods, and illustrate the size-dependent effective shear modulus through some numerical examples. The material constants used in the computation are $\mu_1 = 209$ GPa, $v_1 = 0.17$ and $\mu_2 = 26$ GPa, $v_2 = 0.33$, $l_m = 4$ μm, $\kappa_2 = 26$ GPa. Three different fiber diameters $D = l_m$, $D = 5l_m$ or $D = 100l_m$ are examined. The effective shear modulus as a function of the fiber volume fraction predicted by the two methods are shown respectively in Fig. 2a–c for the three different fiber diameters. The predicted effective shear modulus for the composite with a classical matrix is also included for comparison. As in the classical micromechanics, the generalized self-consistent method leads to a stiffer response than that by Mori–Tanaka's method, which is a lower bound for a classical stiffer fiber reinforced composite. The predicted effective shear modulus as a function of the fiber diameter is illustrated in Fig. 3 for the volume fractions of the fiber $f_1 = 0.2$ and $f_1 = 0.35$ respectively. It is found that when the fiber diameter has the same order as the intrinsic length of the matrix material, the size dependence of the effective shear modulus is more pronounced, and when the fiber diameter becomes very large, the prediction with the proposed model is reduced to the classical result, as required.

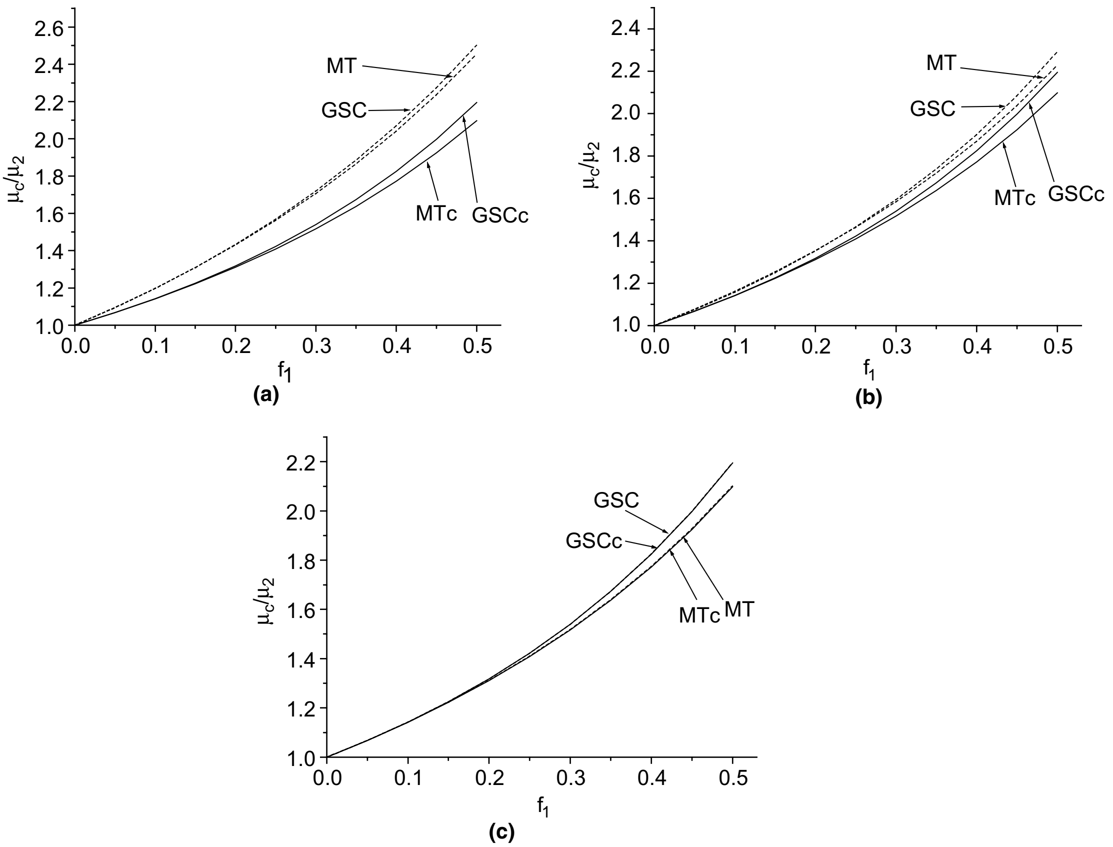

Fig. 2. Effective in-plane shear modulus predicted by Mori–Tanaka's method (MT) and generalized self-consistent method (GSC) as a function of volume fraction of fibers for different fiber sizes: (a) $D = l_m$, (b) $D = 5l_m$, (c) $D = 100l_m$. (MTc, GSCc mean the prediction by Mori–Tanaka's method and by the generalized self-consistent method with a classical matrix.)

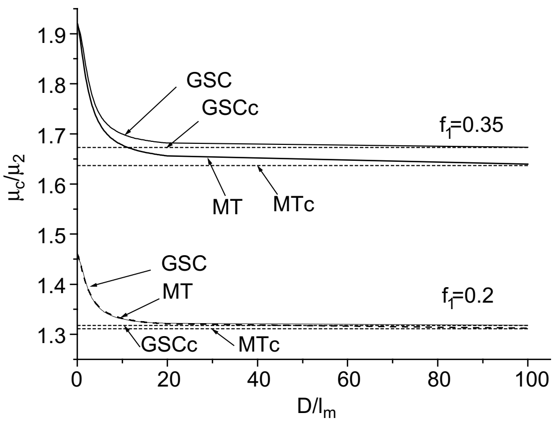

Fig. 3. Effective in-plane shear modulus as a function of fiber diameter predicted by Mori–Tanaka's method and by generalized self-consistent method.

4. Nonlinear in-plane effective property of a micropolar composite

4.1. Second-order stress and couple stress moment in a micropolar matrix

With the help of the effective elastic moduli obtained in Section 3, we will examine the effective nonlinear behavior of a micropolar fiber composite. To this end, we will firstly evaluate the average equivalent stress defined by Eq. (10) for a micropolar matrix in the micropolar fiber composite. With this average equivalent stress and the secant moduli method, the nonlinear behavior of the composite can then be estimated following the same method as for a Cauchy composite (Qiu and Weng, 1992; Suquet, 1995; Hu, 1996, 1997; Hu et al., 1998). This method shares the same structure as Ponte Castañeda's variational method (Ponte Castañeda, 1991) for a Cauchy composite material (Suquet, 1995; Hu, 1996).

We will consider a linear comparison micropolar composite, whose elastic moduli of the matrix are set to be equal to the secant moduli of the actual matrix in the actual composite, defined by Eq. (14). Following the general perturbation method for a micropolar composite recently proposed by Liu and Hu (in press), for a RVE under a constant symmetric macroscopic stress $\bar{\sigma}_{(\alpha\beta)}$ (the macroscopic couple stress is zero), the definition of the micro-macro transition for a general micropolar composite leads to

$$\langle \sigma_{\alpha\beta} m_{\alpha\beta\gamma\lambda} \sigma_{\gamma\lambda} + d m_{\alpha3} m_{\alpha3} \rangle = \bar{\sigma}_{(\alpha\beta)} \overline{M}^{s}_{\alpha\beta\lambda\gamma} \bar{\sigma}_{(\lambda\gamma)} \tag{19}$$

where $\overline{M}^{s}_{\alpha\beta\lambda\gamma}$ is the effective in-plane compliance tensor of the linear comparison composite, $m_{\alpha\beta\gamma\lambda}$ and d is the local in-plane compliances for the stress and the couple stress respectively.

Let the macroscopic stress $\bar{\sigma}_{(\alpha\beta)}$ be kept constant, and the local compliances have variations δd, $\delta m_{\alpha\beta\gamma\lambda}$. This will lead to the variations of the local stress, the couple stress and the effective compliance of the composite. It follows that Eq. (19) becomes

$$\langle \sigma_{\alpha\beta} \delta m_{\alpha\beta\gamma\lambda} \sigma_{\gamma\lambda} + \delta d m_{\alpha3} m_{\alpha3} \rangle + 2\langle \sigma_{\alpha\beta} m_{\alpha\beta\gamma\lambda} \delta\sigma_{\gamma\lambda} + d m_{\alpha3} \delta m_{\alpha3} \rangle = \bar{\sigma}_{(\alpha\beta)} \delta\overline{M}^{s}_{\alpha\beta\lambda\gamma} \bar{\sigma}_{(\lambda\gamma)} \tag{20}$$

It is easy to show that

$$\langle \sigma_{\alpha\beta} m_{\alpha\beta\gamma\lambda} \delta\sigma_{\gamma\lambda} + d m_{\alpha3} \delta m_{\alpha3} \rangle = \langle \sigma_{\alpha\beta} m_{\alpha\beta\gamma\lambda} \rangle \langle \delta\sigma_{\gamma\lambda} \rangle + \langle d m_{\alpha3} \rangle \langle \delta m_{\alpha3} \rangle = 0 \tag{21}$$

Finally we have

$$\langle \sigma_{\alpha\beta} \delta m_{\alpha\beta\gamma\lambda} \sigma_{\gamma\lambda} + \delta d m_{\alpha3} m_{\alpha3} \rangle = \bar{\sigma}_{(\alpha\beta)} \delta\overline{M}^{s}_{\alpha\beta\lambda\gamma} \bar{\sigma}_{(\lambda\gamma)} \tag{22}$$

Now we only let the local matrix compliance have a variation. With the energy density defined by Eq. (7), it has

$$f_2 \left\langle \delta \frac{1}{2\mu_2^{s}} s_{(\alpha\beta)} s_{(\alpha\beta)} + \delta \frac{1}{2\kappa_2} \sigma_{\langle\alpha\beta\rangle} \sigma_{\langle\alpha\beta\rangle} + \delta \frac{1}{k_2} \sigma^2 + \delta \frac{1}{2\beta_2^{s}} m_{\alpha3} m_{\alpha3} \right\rangle_2 = \delta \frac{1}{2\mu_c^{s}} \bar{s}_{(\alpha\beta)} \bar{s}_{(\alpha\beta)} + \delta \frac{1}{k_c^{s}} \bar{\sigma}^2 \tag{23}$$

The superscript 's' means the quantities which are associated with the linear comparison composite. The constants κ_2, k_2 are the same as their elastic cases during the nonlinear deformation, so the superscript 's' is dropped for simplicity. Let the matrix elastic constants (in the comparison composite) μ_2^{s}, κ_2 and β_2^{s} undergo respectively independent variations, $\delta\mu_2^{s}$, $\delta\kappa_2$ and $\delta\beta_2^{s}$, and the other constants remain unchanged. Then we arrive at the following equations:

$$(1 - f_1) \langle s_{(\alpha\beta)} s_{(\alpha\beta)} \rangle_2 \delta(1/2\mu_2^{s}) = \bar{\boldsymbol{\sigma}}^{s} : \delta\overline{\boldsymbol{M}}^{s} : \bar{\boldsymbol{\sigma}}^{s} \tag{24a}$$

$$(1 - f_1) \langle \sigma_{\langle\alpha\beta\rangle} \sigma_{\langle\alpha\beta\rangle} \rangle_2 \delta(1/2\kappa_2) = \bar{\boldsymbol{\sigma}}^{s} : \delta\overline{\boldsymbol{M}}^{s} : \bar{\boldsymbol{\sigma}}^{s} \tag{24b}$$

$$(1 - f_1) \langle m_{\alpha3} m_{\alpha3} \rangle_2 \delta(1/2\beta_2^{s}) = \bar{\boldsymbol{\sigma}}^{s} : \delta\overline{\boldsymbol{M}}^{s} : \bar{\boldsymbol{\sigma}}^{s} \tag{24c}$$

The superscript 's' for the macroscopic stress $\bar{\boldsymbol{\sigma}}$ means that it is a symmetric tensor.

Finally the average equivalent stress of the micropolar matrix in the comparison micropolar composite defined by Eq. (10) can be evaluated by

$$\langle\tilde{\sigma}_e^2\rangle_2 = -\frac{3}{1-f_1}\bar{\boldsymbol{\sigma}}^s : \left[\mu_2^{s^2}\frac{\partial\overline{\boldsymbol{M}}^s}{\partial\mu_2^s} + \frac{\beta_2^{s^2}}{l_m^2}\frac{\partial\overline{\boldsymbol{M}}^s}{\partial\beta_2^s}\right] : \bar{\boldsymbol{\sigma}}^s \tag{25}$$

The effective compliance tensor $\overline{\boldsymbol{M}}^s$ of the linear comparison micropolar composite can be evaluated by the method proposed in Section 3, see Mori–Tanaka's method or the generalized self-consistent method, so the average effective stress of the matrix can then be computed by evaluating Eq. (25), and the nonlinear behavior of the micropolar fiber composite can be analyzed with the aid of the secant moduli method. For the details on the secant moduli method, the readers are referred to the references (Qiu and Weng, 1992; Suquet, 1995; Hu, 1996, 1997; Hu et al., 1998). An analytical initial yield function (in-plane loading) of the micropolar fiber composite can be obtained by setting $\langle\tilde{\sigma}_e^2\rangle_2 = \sigma_y^2$ with the elastic property of the matrix.

4.2. Initial yield function of a micropolar fiber composite

In this section, we will examine the initial yield stress of a micropolar fiber composite and analyze the influence of the fiber diameter on the yield surface of the composite. Since only initial yield stress will be examined, the superscript 's' is dropped in the following formulation. By setting the average equivalent stress defined by Eq. (25) to be the initial yield stress of the matrix σ_y, a general analytical expression of the in-plane yield surface for a micropolar fiber composite can be obtained. Eq. (25) can be further expressed as

$$\langle\tilde{\sigma}_e^2\rangle_2 = \frac{3}{2f_2}\left[\frac{\mu_2^2}{\mu_c^2}\frac{\partial\mu_c}{\partial\mu_2} + \frac{1}{l_m^2}\frac{\beta_2^2}{\mu_c^2}\frac{\partial\mu_c}{\partial\beta_2}\right]\bar{s}_{(\alpha\beta)}\bar{s}_{(\alpha\beta)} + \frac{3}{f_2}\left(\frac{\mu_2^2}{k_c^2}\frac{\partial k_c}{\partial\mu_2}\right)\bar{\sigma}^2 = \frac{\bar{s}_e^2}{A^2} + \frac{\bar{\sigma}^2}{B^2} \tag{26}$$

where $\bar{s}_e^2 = 3\bar{s}_{(\alpha\beta)}\bar{s}_{(\alpha\beta)}/2$. In the derivation of Eq. (26), the relation $\partial k_c/\partial\beta_2 = 0$ has been used, since Mori–Tanaka's method and the generalized self-consistent method show that the effective in-plane bulk modulus is the same as a Cauchy composite.

In order to obtain an analytical expression of the yield function, in the following, we utilize only the effective shear and in-plane bulk moduli estimated by Mori–Tanaka's method. With the help of Eq. (17), we obtain the following analytical expressions of the yield function for a micropolar fiber composite, which are expressed through the non-dimension parameters defined as $\hat{\mu} = \frac{\mu_1}{\mu_2}$, $\hat{k} = \frac{k_1}{k_2}$, $\rho = \frac{\kappa_2}{\mu_2}$, $\chi = \frac{k_1}{\mu_2}$. We let $\hat{k} = \frac{k_1}{k_2} \to 0$ for the elastically incompressible matrix.

(a) In-plane yield function of a general fiber composite

By setting the expressions of the effective shear and in-plane bulk moduli (Eq. (17)) into Eq. (26), we obtain

$$A^2 = \frac{(1-f_1)\left\{1 + [f_1 + 2(1-f_1)\langle K_{1212}^s\rangle_1](\hat{\mu}-1)\right\}^2}{[1+2(1-f_1)\langle K_{1212}^s\rangle_1(\hat{\mu}-1)]^2 + f_1[-1+2(1-f_1)(\langle K_{1212}^s\rangle_1 - ks\mu - ks\beta)(\hat{\mu}-1)^2]} \tag{27a}$$

$$B^2 = \frac{(1-f_1+\chi)^2}{3f_1} \tag{27b}$$

where $ks\mu = \mu_2\frac{\partial\langle K_{1212}^s\rangle_1}{\partial\mu_2}$, $ks\beta = \beta_2\frac{\partial\langle K_{1212}^s\rangle_1}{\partial\beta_2}$.

A size-dependent yield function of the fiber composite is simply written as

$$\frac{\bar{s}_e^2}{A^2} + \frac{\bar{\sigma}^2}{B^2} - \sigma_y^2 = 0 \tag{28}$$

(b) In-plane yield function of porous materials
In this case, letting $\hat{\mu} = 0$, $\chi = 0$, we get

$$A^2 = \frac{(1-f_1)\{1-f_1-2(1-f_1)\langle K^s_{1212}\rangle_1\}^2}{[1-2(1-f_1)\langle K^s_{1212}\rangle_1]^2 + f[-1+2(1-f_1)(\langle K^s_{1212}\rangle_1 - ks\mu - ks\beta)]} \tag{29a}$$

$$B^2 = \frac{(1-f_1)^2}{3f_1} \tag{29b}$$

(c) In-plane yield function of a rigid fiber reinforced composite
In this case, letting $\hat{\mu} \to \infty$, $\chi \to \infty$, we have

$$A^2 = \frac{[f_1 + 2(1-f_1)\langle K^s_{1212}\rangle_1]^2}{4(1-f_1)\langle K^s_{1212}\rangle_1^2 + 2f_1(\langle K^s_{1212}\rangle_1 - ks\mu - ks\beta)} \tag{30a}$$

$$B^2 \to \infty \tag{30b}$$

The influence of the fiber size (diameter) on the initial yield function is described by $\langle K^s_{1212}\rangle_1$ and its derivatives through the parameter $\eta = a/g = \sqrt{2\rho/(1+\rho)}a/l_m$. In what follows, we will present some numerical examples to illustrate the size dependence of the yield stress for fiber composites. The material constants used in the computation are $\mu_1 = 209$ GPa, $v_1 = 0.17$ and $\mu_2 = 26$ GPa, $v_2 = 0.33$, $\kappa_2 = 26$ GPa, $\sigma_y = 250$ MPa. Fig. 4 shows the initial shear yield stress of the composite as a function of the fiber diameter D for two intrinsic lengths of the matrix material. The volume fraction of the fiber is kept to be $f_1 = 0.2$ for the both cases. As expected, the proposed analytical model predicts larger yield stress in shear for the composite with small diameter fibers. The yield surfaces of the composite with different properties of the fiber are shown in Fig. 5, where, for each case, three intrinsic length scales of the matrix are examined, but the material constants used in the computation are the same. It is found that the influence of the fiber size (diameter) on the yield stress is more pronounced for rigid fibers and for shear loading.

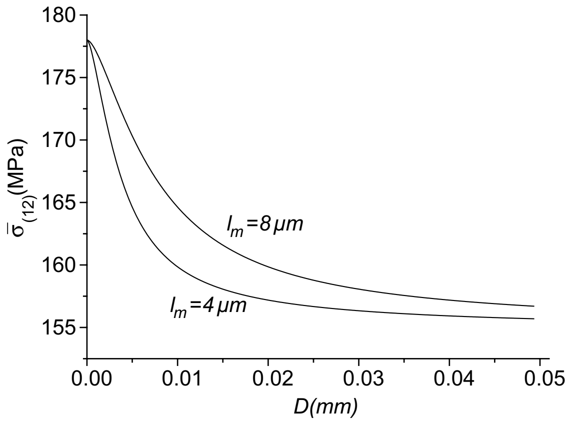

Fig. 4. Predicted shear yield stress as a function of fiber diameter.

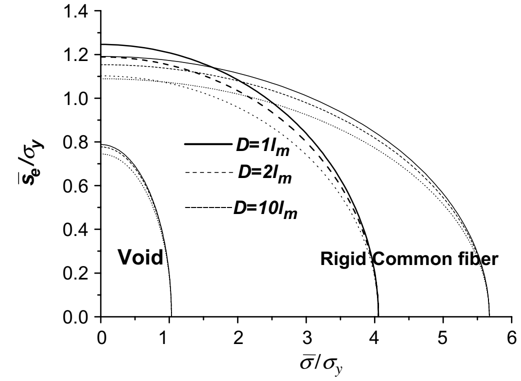

Fig. 5. Yield surfaces for different fiber properties and intrinsic lengths of the matrix.

4.3. Nonlinear in–plane effective shear stress and strain relations

When the applied force is large enough, the matrix material may undergo a plastic deformation. To model the hardening behavior of the composite, the secant moduli method will be utilized. The detailed method for a classical composite can be found in the references (Qiu and Weng, 1992; Suquet, 1995; Hu, 1996, 1997; Hu et al., 1998), and for a micropolar composite, in Liu and Hu (in press). For the classical micromechanics, Suquet (1995) and Hu (1996) have demonstrated that this secant moduli method is equivalent to the variational method proposed by Ponte Castañeda (1991). This in fact provides a new physical interpretation of the secant moduli method in terms of the variational principle. In this section, we will examine the influence of the fiber size on the effective shear stress and strain relation. Assuming a fiber composite is under a shear loading at infinity, the fiber is always elastic. The elastic constants of the fiber and the matrix material are taken to be the same as in Section 3. The material constants for a power type hardening law are respectively $\sigma_y = 250$ MPa, $h = 173$ MPa, and $n = 0.455$.

Fig. 6 shows the predicted effective shear stress and strain relations by the generalized self-consistent method and by Mori–Tanaka's method respectively for different fiber diameters. The volume fraction of

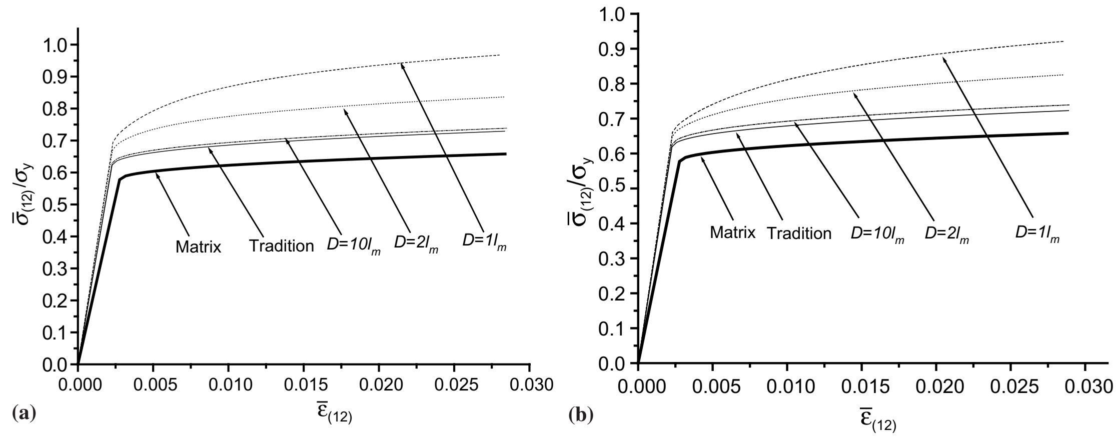

Fig. 6. Comparison of predicted effective shear stress and strain relations by generalized self-consistent method and by Mori–Tanaka's method for different fiber sizes: (a) generalized self-consistent method; (b) Mori–Tanaka's method.

fiber is set to be $f_1 = 0.20$ and $l_m = 4$ μm. The fiber diameter D is chosen to be 1, 2 and 10 times the matrix characteristic length l_m. The shear stress is normalized by the initial yield stress of the matrix, and the predictions with a classical matrix are also included for comparison. As shown in Fig. 6, the proposed method predicts a stronger size dependence of the overall nonlinear shear stress and strain relation when the diameter of fibers approaches to the intrinsic length l_m of the matrix. As expected, when the fiber diameter becomes large, the prediction by the proposed method tends to the classical results for both Mori–Tanaka's method and the generalized self-consistent method. The comparison of Mori–Tanaka's method and the generalized self-consistent method is also given in Fig. 7 for different volume fractions of fibers $f_1 = 0.20, 0.35$, where the prediction by Mori–Tanaka's method with a classical matrix is also included for

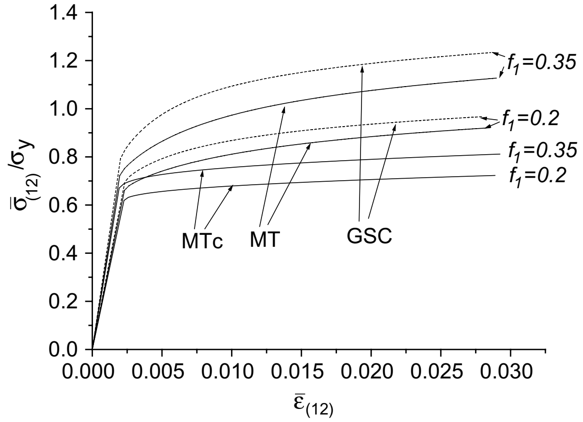

Fig. 7. Comparison of generalized self-consistent method and Mori–Tanaka's method for predicting the effective shear stress and strain relation for a micropolar composite at different volume concentrations of fibers.

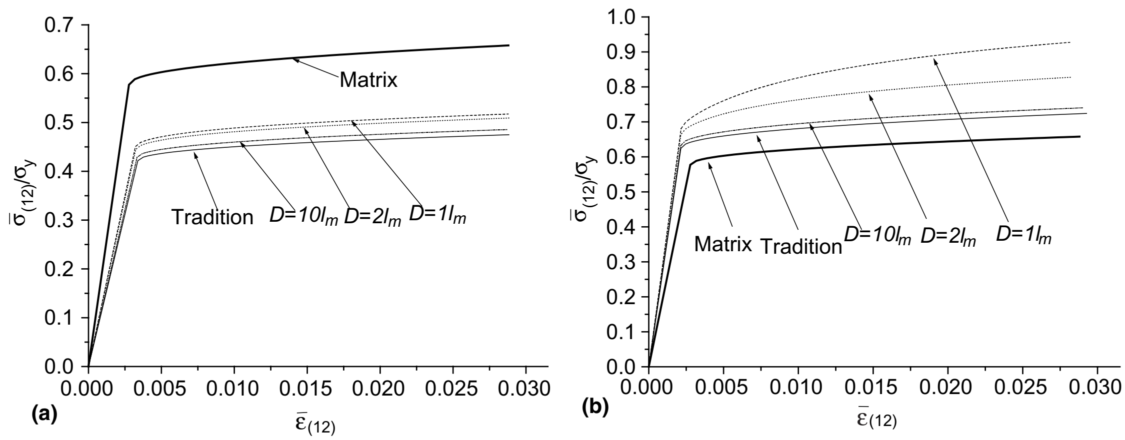

Fig. 8. Predicted effective shear stress and strain relations by extended Mori–Tanaka's method for composites with (a) cylindrical voids and (b) rigid fibers.

comparison (MTc). It is found that the prediction based on the generalized self-consistent method is always stiffer than that based on Mori–Tanaka's method, which is a lower bound for the composite with a classical matrix (the fiber is the harder phase). For a micropolar composite, the bounding nature of the extended Mori–Tanaka's method is not clear, since the Hashin-Shtrikman like bounds for micropolar composites are not available at present.

Finally, the predicted effective shear stress and strain relations by the extended Mori–Tanaka's method for the composite with cylindrical voids or rigid fibers are shown respectively in Fig. 8a and b. As in the case of the initial yield surface of the composite, the size dependence is more pronounced for the composite with rigid fibers, and much less for the material with voids.

5. Conclusion

An analytical micromechanical method is proposed to explain the size dependence of the in-plane effective properties of a fiber composite. The matrix material is idealized as a micropolar material due to its coarse-grain microstructure compared to the size of the reinforced fiber. Mori–Tanaka's method and generalized self-consistent method are extended to a micropolar composite for predicting the effective moduli of the composite. Size-dependent nonlinear effective stress and strain relations of the composite are predicted by extending the secant moduli method based on second-order moment to a micropolar fiber composite. An analytical expression of the size-dependent yield function of a fiber composite is also derived. The results show that the size dependence of the yield surface and that of the nonlinear effective shear stress and strain relation are more pronounced for the composite with hard fibers and for shear loading. When the fiber diameter becomes very large compared to the intrinsic length scale of the matrix, the proposed method reduces to the classical results, as required.

Acknowledgements

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Appendix A. Stress concentration coefficients

For a coated fiber embedded in an infinite material (the three of these materials may be micropolar materials) under remote uniform macroscopic stresses $\bar{s}_{11}$, $\bar{s}_{22}$ and zero couple stress, according to Eringen (1999), the general solution can be obtained by introducing in each region the stress and couple stress potentials F_i and G_i ($i = 1, 2, 3$, it refers to the fiber, the coated layer and the infinite material respectively), and the stress and couple stress are related by the potentials in a cylindrical coordinate system

$$\begin{aligned}
\sigma^i_{rr} &= \frac{1}{r}\frac{\partial F_i}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F_i}{\partial \theta^2} - \frac{1}{r}\frac{\partial^2 G_i}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial G_i}{\partial \theta} \\
\sigma^i_{\theta\theta} &= \frac{\partial^2 F_i}{\partial r^2} + \frac{1}{r}\frac{\partial^2 G_i}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial G_i}{\partial \theta} \\
\sigma^i_{r\theta} &= -\frac{1}{r}\frac{\partial^2 F_i}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial F_i}{\partial \theta} - \frac{1}{r}\frac{\partial G_i}{\partial r} - \frac{1}{r^2}\frac{\partial^2 G_i}{\partial \theta^2} \\
\sigma^i_{\theta r} &= -\frac{1}{r}\frac{\partial^2 F_i}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial F_i}{\partial \theta} + \frac{\partial^2 G_i}{\partial r^2}
\end{aligned} \tag{A.1}$$

$$m^i_{rz} = \frac{\partial G_i}{\partial r}$$

$$m^i_{\theta z} = \frac{1}{r}\frac{\partial G_i}{\partial \theta}$$

The compatibility conditions for each region now become (Eringen, 1999)

$$\begin{aligned} &\frac{\partial}{\partial r}(G_i - c_i^2\nabla^2 G_i) = -2(1-v_i)b_i^2\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2 F_i) \\ &\frac{1}{r}\frac{\partial}{\partial\theta}(G_i - c_i^2\nabla^2 G_i) = 2(1-v_i)b_i^2\frac{\partial}{\partial r}(\nabla^2 F_i) \end{aligned} \tag{A.2}$$

where

$$b_i^2 = \frac{\beta_i}{2\mu_i} = \frac{\kappa_i}{\kappa_i + 2\mu_i}c_i^2 = d_i c_i^2, \quad v_i = \frac{\lambda_i}{2(\lambda_i + \mu_i)} \tag{A.3}$$

Eq. (A.2) leads to the following differential equations for the stress and couple stress potentials:

$$\begin{aligned} &\nabla^4 F_i = 0 \\ &\nabla^2(G_i - c_i^2\nabla^2 G_i) = 0 \end{aligned} \tag{A.4}$$

where ∇^2 is Laplacian operator.

The general solutions of Eq. (A.3) are expressed in the region i as:

$$\begin{aligned} &F_i = A_1^i a^2 \operatorname{Log} r + A_2^i r^2 + (A_3^i a^2 + A_4^i r^2 + A_5^i a^4 r^{-2} + A_6^i a^{-2} r^4)\cos 2\theta \\ &G_i = [A_7^i a^4 r^{-2} + A_8^i r^2 + A_9^i a^2 K_2(r/c_i) + A_{10}^i a^2 I_2(r/c_i)]\sin 2\theta \end{aligned} \tag{A.5}$$

where $I_M(r/c_i)$ is the modified Bessel function of the first kind of order M, $K_M(r/c_i)$ is the modified Bessel function of the second kind of order M, and A_j^i (the superscript $i = 1,2,3$, referring to the different regions, and $j = 1,2,\ldots,10$) are the constants to be determined, a means the fiber radius.

For the problem considered in this paper, the remote boundary condition can be written as $r \to \infty$, $t_{rr}^3 = \frac{1}{2}(\bar{s}_{11} + \bar{s}_{22}) + \frac{1}{2}(\bar{s}_{11} - \bar{s}_{22})\cos 2\theta$, $t_{r\theta}^3 = \frac{1}{2}(\bar{s}_{22} - \bar{s}_{11})\sin 2\theta$, $m_{rz}^3 = 0$, this leads to the following conditions ($R_2 \leqslant r < \infty$, R_2 is the radius of the coated layer):

$$A_6^3 = A_8^3 = A_{10}^3 = 0, \quad A_2^3 = \frac{1}{2}(\bar{s}_{11} + \bar{s}_{22}), \quad A_4^3 = \frac{1}{2}(\bar{s}_{11} - \bar{s}_{22}) \tag{A.6}$$

For the fiber ($0 \leqslant r \leqslant a$), due to the finite stress and couple stress, we have

$$A_1^1 = A_3^1 = A_5^1 = A_7^1 = A_9^1 = 0 \tag{A.7}$$

The other unknown constants can be determined from the continuity conditions at the interface between the fiber and the coated layer, and the interface between the coated layer and the infinite material, which are written as (for $i = 1,2$)

$$\begin{aligned} &u_r^i(R_i) = u_r^{i+1}(R_i), \quad u_\theta^i(R_i) = u_\theta^{i+1}(R_i), \quad t_r^i(R_i) = t_r^{i+1}(R_i), t_{r\theta}^i(R_i) = t_{r\theta}^{i+1}(R_i) \\ &m_{rz}^i(R_i) = m_{rz}^{i+1}(R_i), \quad \varphi_z^i(R_i) = \varphi_z^{i+1}(R_i) \end{aligned} \tag{A.8}$$

where $R_1 = a$.

The above condition can be rewritten in a more compact form, if we note

$$\begin{aligned}
\overline{A}^1 &= \{0, A_2^1, 0, A_4^1, 0, A_6^1, 0, A_8^1, 0, A_{10}^1\}^{\mathrm{T}} \\
\overline{A}^2 &= \{A_1^2, A_2^2, A_3^2, A_4^2, A_5^2, A_6^2, A_7^2, A_8^2, A_9^2, A_{10}^2\}^{\mathrm{T}} \\
\overline{A}^3 &= \{A_1^3, A_2^3, A_3^3, A_4^3, A_5^3, 0, A_7^3, 0, A_9^3, 0\}^{\mathrm{T}}
\end{aligned} \tag{A.9}$$

Eq. (A.8) together with the detailed expressions for the local fields, and the condition (A.5) and (A.7) can be written together in a compact form as $(i = 1, 2)$

$$M_i(R_i)\overline{A}^i = M_{i+1}(R_i)\overline{A}^{i+1} \tag{A.10}$$

The expression for M_i will be listed in the end of Appendix A.

For a remote in-plane hydrostatic loading, $\bar{s}_{11} = \bar{s}_{22} = \bar{\sigma}$, the average stresses in the fiber and the coated layer are expressed as

$$\langle \sigma \rangle_i = 2A_2^i = s_i \bar{\sigma} \tag{A.11}$$

For a remote shear loading, $\bar{s}_{11} = -\bar{s}_{22}$, after a lengthy mathematical manipulation, we get finally the average stress and couple stress in the fiber and the coated layer

$$\langle s_{(\alpha\beta)} \rangle = -(2A_4^1 + 2A_8^1 + 3A_6^1) - I_1(a/c_1)aA_{10}^1/2c_1 = p_1\bar{s}_{(\alpha\beta)} \tag{A.12}$$

$$\begin{aligned}
\langle s_{(\alpha\beta)} \rangle_2 = -[2A_4^2 + 3(1+t^2)A_6^2 + 2A_8^2] - \frac{a}{2c_2(t^2-1)}\{[-K_1(a/c_2) + tK_1(at/c_2)]A_9^2 \\
+ [I_1(a/c_2) - tI_1(at/c_2)]A_{10}^2\} = p_2\bar{s}_{(\alpha\beta)}
\end{aligned} \tag{A.13}$$

where $t = R_2/a$, the stress concentration coefficients p_1, s_1, p_2, s_2 are determined through the constants A_j^i, which are determined by Eq. (A.10). The expression for the matrix $M_i(r)$ is

$$\begin{bmatrix}
\frac{a^2}{r^2}, 2, 0, 0, 0, 0, 0, 0, 0, 0 \\
\frac{-a^2}{2\mu_2 r}, \frac{r}{\mu_2 + \lambda_2}, 0, 0, 0, 0, 0, 0, 0, 0 \\
0, 0, -\frac{4a^2}{r^2}, -2, -\frac{6a^4}{r^4}, 0, \frac{6a^4}{r^4}, -2, \frac{6a^2}{r^2}K_0\left(\frac{r}{c_i}\right) + \left(\frac{12c_i a^2}{r^3} + \frac{2a^2}{c_i r}\right)K_1\left(\frac{r}{c_i}\right), \frac{6a^2}{r^2}I_0\left(\frac{r}{c_i}\right) - \left(\frac{12c_i a^2}{r^3} + \frac{2a^2}{c_i r}\right)I_1\left(\frac{r}{c_i}\right) \\
0, 0, -\frac{2a^2}{r^2}, 2, -\frac{6a^4}{r^4}, \frac{6r^2}{a^2}, \frac{6a^4}{r^4}, 2, \frac{6a^2}{r^2}K_0\left(\frac{r}{c_i}\right) + \left(\frac{12c_i a^2}{r^3} + \frac{2a^2}{c_i r}\right)K_1\left(\frac{r}{c_i}\right), \frac{6a^2}{r^2}I_0\left(\frac{r}{c_i}\right) - \left(\frac{12c_i a^2}{r^3} + \frac{2a^2}{c_i r}\right)I_1\left(\frac{r}{c_i}\right) \\
0, 0, 0, 0, 0, 0, -\frac{2a^4}{r^3}, 2r, -\frac{2a^2}{r}K_0\left(\frac{r}{c_i}\right) - \left(\frac{4c_i a^2}{r^2} + \frac{a^2}{c_i}\right)K_1\left(\frac{r}{c_i}\right), -\frac{2a^2}{r}I_0\left(\frac{r}{c_i}\right) + \left(\frac{4c_i a^2}{r^2} + \frac{a^2}{c_i}\right)I_1\left(\frac{r}{c_i}\right) \\
0, 0, \frac{a^2(\lambda_i + 2\mu_i)}{r\mu_i(\lambda_i + \mu_i)}, -\frac{r}{\mu_i}, \frac{a^4}{r^3\mu_i}, -\frac{r^3\lambda_i}{a^2\mu_i(\lambda_i + \mu_i)}, -\frac{a^4}{r^3\mu_i}, -\frac{r}{\mu_i}, -\frac{a^2}{r\mu_i}K_0\left(\frac{r}{c_i}\right) - \frac{2c_i a^2}{r^2\mu_i}K_1\left(\frac{r}{c_i}\right), -\frac{a^2}{r\mu_i}I_0\left(\frac{r}{c_i}\right) + \frac{2c_i a^2}{r^2\mu_i}I_1\left(\frac{r}{c_i}\right) \\
0, 0, \frac{-a^2}{r\mu_i(\lambda_i + \mu_i)}, \frac{r}{\mu_i}, \frac{a^4}{r^3\mu_i}, \frac{r^3(2\lambda_i + 3\mu_i)}{a^2\mu_i(\lambda_i + \mu_i)}, -\frac{a^4}{r^3\mu_i}, \frac{r}{\mu_i}, -\frac{a^2}{r\mu_i}K_0\left(\frac{r}{c_i}\right) - \frac{(4c_i^2 + r^2)a^2}{2r^2 c_i\mu_i}K_1\left(\frac{r}{c_i}\right), -\frac{a^2}{r\mu_i}I_0\left(\frac{r}{c_i}\right) + \frac{(4c_i^2 + r^2)a^2}{2r^2 c_i\mu_i}I_1\left(\frac{r}{c_i}\right) \\
0, 0, 0, 0, 0, 0, \frac{a^4}{r^2\beta_i}, \frac{r^4}{\beta_i}, \frac{a^2}{\beta_i}K_0\left(\frac{r}{c_i}\right) + \frac{2c_i a^2}{r\beta_i}K_1\left(\frac{r}{c_i}\right), \frac{a^2}{\beta_i}I_0\left(\frac{r}{c_i}\right) - \frac{2c_i a^2}{r\beta_i}I_1\left(\frac{r}{c_i}\right)
\end{bmatrix}$$

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