Design of arbitrary shaped pentamode acoustic cloak based on quasi-symmetric mapping gradient algorithm

Yi Chen, Xiaoning Liu, and Gengkai Hu[a)]
School of Aerospace Engineering, Beijing Institute of Technology, Beijing 100081, China
chenyi221@gmail.com, liuxn@bit.edu.cn, hugeng@bit.edu.cn

Abstract: Due to solid and broadband nature, pentamode acoustic cloak is more promising for engineering applications. A simple algorithm based on an elasticity equation is proposed to obtain quasi-symmetric mapping gradient and in turn the characteristic stress for arbitrary shape cloaks. A high degree of symmetry of the obtained mapping gradient and nearly perfect cloaking effect of the designed pentamode cloaks are confirmed by numerical examples. The proposed method paves the way to design more complicated transformation devices with pentamode materials.

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1. Introduction

Rendering an object invisible by designing a coating layer is a long standing inverse problem. In 2006, based on the property of form invariance of Maxwell's equation under coordinate mapping, a concise mathematical approach was proposed to design material distribution for a targeted wave pattern.[1,2] This method usually called transformation electromagnetics was soon validated by experiment through making an invisible cloak realized with a metamaterial technique.[3] Transformation acoustics based on meta-fluid of anisotropic density was first proposed by observing the similarity between acoustic equation and Maxwell's equation.[4] Meta-fluids with anisotropic density can only be realized using metamaterial technology, such as alternating different fluid layers,[5] and they are basically fluidic in nature.

Another route is to make use of meta-fluids with anisotropic modulus, i.e., pentamode (PM) materials, which can be completely constructed from solids through careful microstructure design.[6] Wave equation for PM materials is found to be form-invariant under a general coordinate mapping, so it can be used to design acoustic cloak with PM materials.[7] These findings stimulate an active study recently for this material conceived theoretically 2 decades ago.[8–12] PM material, including conventional fluid as a special case, is defined as an elastic material with only one non-zero eigenvalue of elasticity matrix, and supports only one kind of stress state. By microstructure design, PM material can be made anisotropic in contrast to conventional fluids.[10,11] In addition, its effective property is obtained under a long wavelength condition without using a resonant mechanism and thus is basically broadband. Acoustic cloak using PM materials has advantages of finite mass, broadband efficiency, and solid nature,[7] is more promising for practical applications. However, in the design of acoustic cloak using PM material, a divergence free and symmetric second order tensor $\mathbf{S}$ is introduced and its choice is not available for general cases. This limits the design of transformation devices with PM material to only simple cases, i.e., cylindrical cloak in two dimensions or spherical cloak in three dimensions.

In this paper, we will re-examine transformation acoustics based on PM material using an operator mapping technique and rigorously derive the continuity condition for the mapped physical field. It is found that the $\mathbf{S}$ tensor must have principal axes parallel or normal to the outer boundary of cloak with a unit normal component in order to guarantee an impedance match at the PM/fluid interface. This additional condition together with divergence free and symmetric property can be fulfilled by taking $\mathbf{S} = J^{-1}\mathbf{F}$ if the mapping gradient $\mathbf{F}$ is symmetric. Based on this observation, a simple numerical method based on the elasticity equation is proposed to derive quasi-symmetric mapping gradient for arbitrary cloaks and to design more complex transformation devices with PM materials.

[a)]Author to whom correspondence should be addressed.

2. Methods

The elastic tensor of PM material is characterized by $\mathbf{C} = K\mathbf{S}\otimes\mathbf{S}$,[6] and $\mathbf{S}$ is called its characteristic stress tensor. The stress in PM material is always proportional to $\mathbf{S}$, i.e., $\boldsymbol{\sigma} = -p\mathbf{S}$, and p is called the pseudo pressure compared to pressure of fluid.[7] Actually conventional fluid can be considered as a natural PM material with the characteristic stress tensor $\mathbf{S} = \mathbf{I}$. The wave equation for PM materials directly follows that of conventional solids with a specified form of stress and modulus, $\dot{p} = K\mathbf{S} : \nabla\mathbf{v}$, $\dot{\mathbf{v}} = -\boldsymbol{\rho}^{-1}\cdot\nabla\cdot(p\mathbf{S})$. Here, an anisotropic dynamic density tensor $\boldsymbol{\rho}$ is assumed in order to cover the general cases. In principal coordinate of the PM material, the characteristic tensor $\mathbf{S}$ is able to write in a simple form, $\mathbf{S} = \sqrt{K_n/K}\mathbf{e}_n\mathbf{e}_n + \sqrt{K_\tau/K}\mathbf{e}_\tau\mathbf{e}_\tau$, in which K_n and K_τ are called two principal bulk moduli of the PM material in contrast to one of the conventional fluids.

A basic idea of transformation acoustics based on PM material is depicted in Fig. 1. A physical space, consisting of three adjacent domains denoted by ω^{out}, ω, and ω^{in}, respectively, is shown in Fig. 1(b). The background domain ω^{out} is occupied by a homogeneous fluid with density ρ_0 and bulk modulus K_0. To derive material property distribution in ω, we consider another virtual space, shown in Fig. 1(a), with two adjacent domains Ω and Ω^{out}. Both of the domains are filled with the same fluid as that in the domain ω^{out} of the physical space. In addition, the outer boundary of the domain Ω and domain ω are exactly the same, i.e., $\partial\Omega = \partial\omega^{+}$. For simplification, two Descartes coordinates are employed to characterize the different locations in the physical and virtual spaces. In the virtual space, the governing equations for acoustic pressure are

$$\dot{\mathbf{v}}(\mathbf{X}) = -\rho_0^{-1}\nabla_{\mathbf{X}}p(\mathbf{X}), \quad \dot{p}(\mathbf{X}) = -K_0\nabla_{\mathbf{X}}\cdot\mathbf{v}(\mathbf{X}). \tag{1}$$

Here $\mathbf{v}(\mathbf{X})$ and $p(\mathbf{X})$ are particle velocity and pressure at location $\mathbf{X}$ in the virtual space, and the symbol $\nabla_{\mathbf{X}}$ represents the gradient operator in the virtual space and the dot over the velocity and pressure symbols stands for derivation with respect to time. Now consider an imaginary mapping function of the locations x and X, $\mathbf{x} = \mathbf{x}(\mathbf{X})$, which maps the domain Ω^{out} onto the domain ω^{out}, and the domain Ω onto ω, respectively. The mapping gradient is defined by $\mathbf{F} = \boldsymbol{\partial}\mathbf{x}/\partial\mathbf{X}$ or in component form $F_{ij} = \partial x_i/\partial X_j$, it is an identity matrix in the domain Ω^{out}, $\mathbf{F} = \mathbf{I}$. Because of the relation between $\mathbf{x}$ and $\mathbf{X}$, one has the following relations between operators $\nabla_{\mathbf{X}} = \mathbf{F}^{\text{T}}\cdot\nabla_{\mathbf{x}}$ and $\nabla_{\mathbf{X}}\cdot = J\nabla_{\mathbf{x}}\cdot(J^{-1}\mathbf{F}\cdot)$, in which the identity $\nabla_{\mathbf{x}}\cdot(J^{-1}\mathbf{F}) = 0$ has been used and J is the Jacobian of $\mathbf{F}$. Therefore, Eq. (1) can be naturally re-casted in terms of $\mathbf{x}$,

$$\dot{p}(\mathbf{X}) = -K_0J\nabla_{\mathbf{x}}\cdot(J^{-1}\mathbf{S}\cdot\mathbf{S}^{-1}\cdot\mathbf{F}\cdot\mathbf{v}(\mathbf{X})) = -K_0J\mathbf{S} : \nabla_{\mathbf{x}}(J^{-1}\mathbf{S}^{-1}\cdot\mathbf{F}\cdot\mathbf{v}(\mathbf{X})), \tag{2a}$$

$$J^{-1}\mathbf{S}^{-1}\cdot\mathbf{F}\cdot\dot{\mathbf{v}}(\mathbf{X}) = -(\rho_0^{-1}J^{-1}\mathbf{S}^{-1}\cdot\mathbf{F}\cdot\mathbf{F}^{\text{T}}\cdot\mathbf{S}^{-1})\cdot\nabla_{\mathbf{x}}(p(\mathbf{X})\mathbf{S}). \tag{2b}$$

A divergence free and symmetric second order tensor in the physical space is introduced here, $\mathbf{S}^{\text{T}} = \mathbf{S}$, $\nabla_{\mathbf{x}}\cdot\mathbf{S} = \mathbf{0}$. An additional term $J^{-1}\mathbf{S}^{-1}\cdot\mathbf{F}$ is multiplied to both sides of the first equation in Eq. (1) on the left. By defining the following new field variables and material property, $\mathbf{v}'(\mathbf{x}) = J^{-1}\mathbf{S}^{-1}\cdot\mathbf{F}\cdot\mathbf{v}(\mathbf{X})$, $p'(\mathbf{x}) = p(\mathbf{X})$, $\boldsymbol{\rho}'^{-1}(\mathbf{x}) = \rho_0^{-1}J^{-1}\mathbf{S}^{-1}\cdot\mathbf{F}\cdot\mathbf{F}^{\text{T}}\cdot\mathbf{S}^{-1}$, $\mathbf{C}'(\mathbf{x}) = K'\mathbf{S}\otimes\mathbf{S}$, $K' = K_0J$. Equations (2) become

$$\dot{p}'(\mathbf{x}) = -K'(\mathbf{x})\mathbf{S} : \nabla_{\mathbf{x}}(\mathbf{v}'(\mathbf{x})), \ \mathbf{v}'(\mathbf{x}) = -\boldsymbol{\rho}'^{-1}(\mathbf{x})\cdot\nabla_{\mathbf{x}}(p'(\mathbf{x})\mathbf{S}). \tag{3}$$

These equations are exactly the same as the wave equation for a general PM material. Thus, if PM materials with the property given by $\boldsymbol{\rho}^{-1}(\mathbf{x})$, $\mathbf{C}(\mathbf{x})$ are distributed in the domain ω, the physical field in the domain $\omega\&\omega^{\text{out}}$ can be directly mapped from that

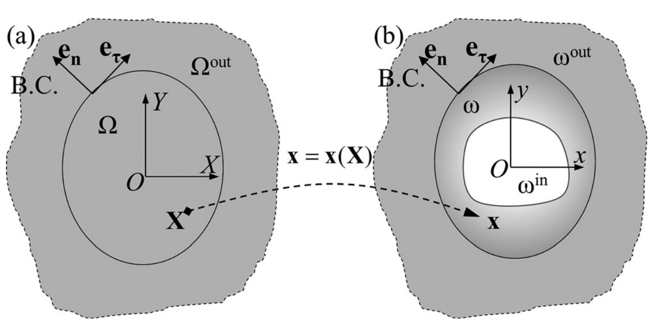

Fig. 1. Illustration of transformation acoustics based on PM material. (a) Virtual space with coordinate XOY. (b) Physical space with coordinate xoy.

in the domain $\Omega^{\&}\Omega^{\text{out}}$ using $\mathbf{v}'(\mathbf{x}) = J^{-1}\mathbf{S}^{-1}\cdot\mathbf{F}\cdot\mathbf{v}(\mathbf{X})$, $p'(\mathbf{x}) = p(\mathbf{X})$. Since no scattering takes place for any incident wave in Ω^{out} due to its homogeneous feature, there will also be no scattering in ω^{out} for the same incident wave. Equations (3) are first derived by Norris[7] in a different way.

In order to achieve a perfect cloaking effect, the mapped solution should also satisfy continuous condition at the boundary. In most cases, this condition for the mapped solution is automatically verified with the continuous condition in the virtual space. As for the considered transformation acoustics based on PM material, the mapped material property is not able to satisfy the continuous condition at the interface $\partial\omega^{+}$ in the physical space, and further analysis is needed.

Since the mapping function should maintain the boundary $\partial\Omega$ unaltered, the mapping gradient $\mathbf{F}$ at the boundary $\partial\Omega = \partial\omega^{+}$ should have the following form: $\mathbf{F} = \mathbf{e}_\tau\mathbf{e}_\tau + (J\mathbf{e}_\text{n} + \alpha\mathbf{e}_\tau)\mathbf{e}_\text{n}$, where $\mathbf{e}_\tau$ and $\mathbf{e}_\text{n}$ are the tangential and normal unit vectors at the boundary, J characterizes the normal extension, and α stands for the distortion. At the boundary $\partial\omega^{+}$, the mapped solution should satisfy the continuity conditions for normal particle velocity and normal stress, respectively. As for the normal stress at the interface $\partial\omega^{+}$, we need

$$[\mathbf{e}_\text{n}\cdot\boldsymbol{\sigma}'(\mathbf{x})]_{|\partial\omega} = -p(\mathbf{X})[\mathbf{e}_\text{n}\cdot\mathbf{S}]_{|\partial\Omega} = 0. \tag{4}$$

Here, the symbol $[\cdot]_{|\partial B}$ denotes the jump of the physical variable across the interface ∂B, and the pressure $p(\mathbf{X})$ is taken out of the jump symbol due to its continuity in the virtual space. Equation (4) provides a constraint condition for $\mathbf{S}$ at the boundary, i.e., $[\mathbf{e}_\text{n}\cdot\mathbf{S}]_{|\partial\Omega} = 0$. Taking into account the symmetry of $\mathbf{S}$ and $\mathbf{S} = \mathbf{I}$ in ω^{out}, we conclude that $\mathbf{S}$ must have principal axes parallel or normal to the boundary $\partial\omega^{+}$, in addition its normal component must be unity. As for the continuity condition of the normal velocity at the interface $\partial\omega^{+}$, we get

$$[\mathbf{e}_\text{n}\cdot\boldsymbol{v}']_{|\partial\omega} = [J^{-1}\mathbf{e}_\text{n}\cdot\mathbf{S}^{-1}\cdot\mathbf{F}\cdot\boldsymbol{v}]_{|\partial\Omega} = 0. \tag{5}$$

Substituting $\mathbf{S}$ and $\mathbf{F}$ into Eq. (5), the continuity condition of the normal velocity is automatically satisfied. So in addition to symmetry and divergence free, the characteristic stress should be $\mathbf{S} = \mathbf{e}_\text{n}\mathbf{e}_\text{n} + S_{\tau\tau}\mathbf{e}_\tau\mathbf{e}_\tau$ at the boundary $\partial\omega^{+}$ to maintain the impedance match condition at the PM/fluid interface $\partial\omega^{+}$. With similar analysis, one can verify that the transformed material properties derived from transformation acoustic with meta-fluid of anisotropic density or transformation optics and asymmetric transformation elasticity automatically satisfy the impedance match conditions.

Because of the stringent constraint on the characteristic stress tensor $\mathbf{S}$, no general method for determining $\mathbf{S}$ is available at present. This is a major obstacle preventing PM materials from practical design for a complex shaped target. One solution is to find a symmetric gradient mapping field, then we can simply take $\mathbf{S}$ as $J^{-1}\mathbf{F} = \mathbf{e}_\text{n}\mathbf{e}_\text{n} + J^{-1}\mathbf{e}_\tau\mathbf{e}_\tau$ in the design of an arbitrary shaped PM cloak. This special form of $\mathbf{S}$ obviously guarantees the impedance match condition at the boundary and makes the required PM material with isotropic density as well, which is preferred in designing PM acoustic cloaks,[11] $\boldsymbol{\rho}'^{-1}(\mathbf{x}) = \rho_0^{-1}J^{-1}\mathbf{S}^{-1}\cdot\mathbf{F}\cdot\mathbf{F}^{\text{T}}\cdot\mathbf{S}^{-1} = \rho_0^{-1}J\mathbf{I}$.

The mapping function is able to choose freely so long as the boundary $\partial\Omega$ remains unaltered, thus it is possible to propose a numerical algorithm to obtain a symmetric or quasi-symmetric mapping gradient, just as the quasi-conformal mapping algorithm widely used in transformation optics.[13] In the following, we will explain how to find an almost completely symmetric mapping gradient for arbitrary shape cloak. Inverse mapping function of $\mathbf{x} = \mathbf{x}(\mathbf{X})$ is denoted as $\mathbf{X} = \mathbf{x} + \mathbf{u}(\mathbf{x})$, which means a point $\mathbf{x}$ in the physical space is displaced to the position $\mathbf{X}$ in the virtual space through elastic displacement $\mathbf{u}$. The displacement should satisfy the following boundary conditions deduced from the mapping operation:

$$\mathbf{u} = 0 \;\; (x,y)\in\partial\omega^{+}, \quad \mathbf{u} = (\varepsilon - 1)\mathbf{x} \;\; (x,y)\in\partial\omega^{-}, \tag{6}$$

where, as frequently used in the transformation method, a positive small parameter $0 < \varepsilon \ll 1$ is introduced to avoid singularity in material property. It can be verified; the displacement must satisfy the differential equation, $\nabla\times\mathbf{u} = \mathbf{0}$, if $\mathbf{F}$ is a symmetric tensor. This first order differential equation is not sufficient to determine the displacement field with the boundary conditions given by Eq. (6). Taking the curl operation on equation $\nabla\times\mathbf{u} = \mathbf{0}$ leads to the following second order differential equation:

$$\nabla\times(\nabla\times\mathbf{u}) = \nabla(\nabla\cdot\mathbf{u}) - \nabla^2\mathbf{u} = 0. \tag{7}$$

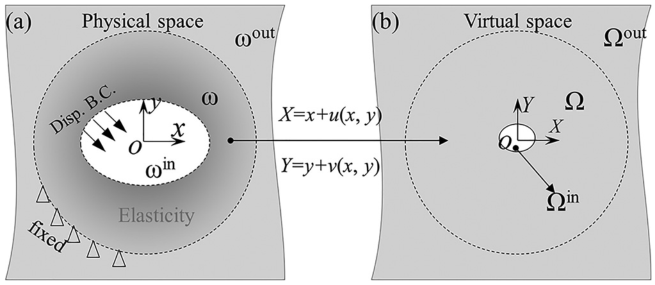

Fig. 2. Design of arbitrary shaped PM acoustic cloak. (a) Physical space with background region ω^{out}, cloaking layer ω, and cloaking region ω^{in}. (b) Virtual space.

This equation is similar to the elasticity equation with special isotropic elastic materials, and can be solved easily using an elasticity module in FEM software with the following special elastic constants:

$$\lambda = (\xi - 2)\mu, \quad |\xi| \ll 1, \tag{8}$$

where another small parameter $|\xi| \ll 1$ is introduced to avoid degeneracy of the elastic equation. With these Lamé constants, the simplified elastic equation can be written as

$$\nabla \times (\nabla \times \mathbf{u}) = \xi \nabla(\nabla \cdot \mathbf{u}). \tag{9}$$

It is seen that Eq. (9) is a good approximation of Eq. (7) when ξ is small enough. Although Eq. (7) does not guarantee $\nabla \times \mathbf{u} = \mathbf{0}$ in reverse, it will be shown by subsequent numerical examples that the displacement field based on Eq. (9) is almost curl-free and thus the mapping gradient $\mathbf{F}$ has a high degree of symmetry. It can also be proved that, for the derived inverse mapping gradient $\mathbf{F}^{-1}$ from Eq. (9) combined with the boundary condition Eq. (6), its overall deviation of symmetry in the domain ω is zero, $\iint_\omega ((\mathbf{F}^{-1})_{21} - (\mathbf{F}^{-1})_{12}) da = \iint_\omega (\nabla \times \mathbf{u}) \cdot da = -(\varepsilon - 1)\oint_{\partial\omega^-} \mathbf{x} \cdot d\mathbf{r} = 0$. After getting the mapping gradient $\mathbf{F}$, one just needs to choose $\mathbf{S}$ as $J^{-1}\mathbf{F}$ and then derives the required PM material property $\boldsymbol{\rho}^{-1}(\mathbf{x})$, $\mathbf{C}(\mathbf{x})$ for the cloaking layer ω.

3. Results and discussion

We consider in the following a physical space and the corresponding virtual space as shown in Fig. 2. In the physical space, the outer boundary of the cloaking layer ω is a circle with radius $r = 2$ m, and its inner boundary is an ellipse with long axis $a_{in} = 1$ m and short axis $b_{in} = a_{in}/1.5$, respectively. The fluid in the background domain has a density $\rho_0 = 1000\,\text{kg/m}^3$ and bulk modulus $K_0 = 2.25$ GPa. The mapping gradient is obtained by solving Eq. (9) with the appropriate boundary condition. Parameters are chosen as $\mu = 1$ Pa, $\varepsilon = 10^{-3}$, and $\xi = 10^{-4}$. Symmetry deviation $|F_{12} - F_{21}|$ shown in Fig. 3(b) is much smaller than 10^{-3} in the most region and becomes a little bit larger only near the inner boundary due to large deformation.

Therefore, when the determined mapping gradient $\mathbf{F}$ shows a high degree of symmetry, we can use $\mathbf{S} = J^{-1}\mathbf{F}$ to obtain the required PM material parameter of the cloak. It is found that the density in the cloaking layer follows $0 < \rho_s/\rho_0 < 2$, and the anisotropy of PM modulus, i.e., the ratio of two principal bulk moduli, is relatively

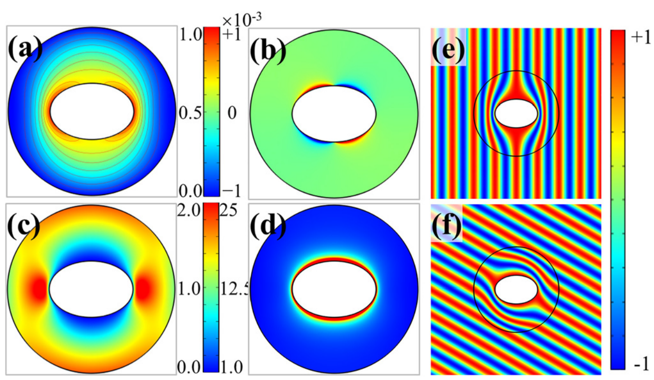

Fig. 3. (Color online) (a) Displacement field $|\mathbf{u}|$. (b) Symmetry plot of $\mathbf{F}$ $(F_{12} - F_{21})$. (c) Normalized density. (d) Anisotropy degree K_t/K_n. (e) and (f) Pressure field for 0° and 45° incident with frequency $f = 1.5$ kHz.

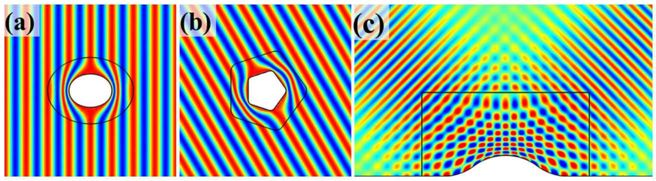

Fig. 4. (Color online) Pressure field for plane wave/Gauss beam incident onto (a) double ellipse, (b) double pentagon, and (c) carpet PM cloak.

small $1 < K_t/K_n < 25$, which lies in the obtainable range of possible microstructure for PM unit cells.[11]

To validate the cloaking effect with the derived PM material property, a plane wave incident onto the designed cloak is simulated using acoustic/solid coupling module in COMSOL Multiphysics. The PM material domain is modelled with an elastic solid and the background domain is modelled using acoustic media with Perfectly Matched Layer enclosing the computed domain. Figures 3(e) and 3(f) show the pressure fields for the plane wave's incident from 0° and 45° directions onto a rigid scatter with the designed cloaking layer, respectively. In the figure, color in the PM material stands for the pseudo pressure $p = -J\sigma_x/F_{xx}$. It is clearly seen that the incident waves onto the cloak are successfully guided by the graded PM material layer to go around the central cloaking region, and come out unperturbed from the other side of the cloak. The proposed numerical method is also applied with success design to complex shaped cloaks, like double ellipse, double pentagon or carpet cloaks, as indicated in Figs. 4(a)–4(c).

4. Conclusions

In conclusion, transformation acoustics based on PM materials is re-examined with emphasis on the continuity condition at boundary of different media. It is found that if the mapping gradient $\mathbf{F}$ is symmetric, the characteristic stress in the form of $\mathbf{S} = J^{-1}\mathbf{F}$ satisfies a naturally continuity condition and symmetric and divergence free as well. An efficient numerical algorithm based on an elasticity equation with special Lamé constants is proposed to obtain a quasi-symmetric mapping gradient, and are further exploited to design arbitrary shaped PM acoustic cloaks. Numerical examples are provided to demonstrate the efficiency of the proposed algorithm, and the cloaking effect of the designed PM acoustic cloaks is also validated through numerical simulation. The proposed method paves the way to design complex shaped acoustic cloaks with PM materials.

Acknowledgments

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